A generalization of Kawanaka's identity for Hall-Littlewood polynomials and applications

Masao Ishikawa[1], Frédéric Jouhet[2] and Jiang Zeng[2,3]

October 14, 2004

[1] Department of Mathematics, Faculty of Education
Tottori University, Tottori 680-8551, Japan,
ishikawa@fed.tottori-u.ac.jp

[2] Institut Girard Desargues, Université Claude Bernard (Lyon I)
43, bd du 11 Novembre 1918, 69622 Villeurbanne Cedex, France
jouhet@igd.univ-lyon1.fr
and
[3] Center for Combinatorics, LPMC, Nankai University
Tianjin 300071, People's Republic of China
zeng@igd.univ-lyon1.fr

Abstract

An infinite summation formula of Hall-Littlewood polynomials due to Kawanaka is generalized to a finite summation formula, which implies, as applications, twelve multiple q-identities of Rogers-Ramanujan type.

1 Introduction

Recently, starting from two infinite summation formulae for Hall-Littlewood polynomials, two of the present authors [5] have generalized a method due to Macdonald [7] to obtain new finite summation formulae for these polynomials, which permit them to extend Stembridge's approach [9] to cover more multiple q-series identities of Rogers-Ramanujan type. Conversely these symmetric functions identities can be viewed as a generalization of

Rogers-Ramanujan identities. In view of the numerous formulae of Rogers-Ramanujan type [8] one can think that there should be more such symmetric functions identities. However there are only few known infinite summation formulae for Hall-Littlewood polynomials, moreover, as pointed out in [5], such kind of generalizations require also the ingeniousness to find the suitable coefficient in order to obtain some reasonably simple and useful formulae.

The purpose of this paper is to show that Kawanaka's recent infinite summation of Hall-Littlewood polynomials is suitable for such an extension. To state his identity we need to recall some standard notations of q-series, as in [3].

Set $(x)_0 := (x;q)_0 = 1$ and for $n \geq 1$

$$(x)_n := (x;q)_n = \prod_{k=0}^{n}(1-xq^k), \quad (x)_\infty := (x;q)_\infty = \prod_{k=0}^{\infty}(1-xq^k).$$

For $n \geq 0$ and $r \geq 1$, set

$$(a_1, \cdots, a_r;\, q)_n = \prod_{i=1}^{r}(a_i)_n, \qquad (a_1, \cdots, a_r;\, q)_\infty = \prod_{i=1}^{r}(a_i)_\infty.$$

The q-binomial identity [1] reads then as follows :

$$\sum_{m\geq 0} \frac{(a)_m}{(q)_m} x^m = \frac{(ax)_\infty}{(x)_\infty}, \tag{1}$$

which reduces to the finite q-binomial identity by substitution $a \leftarrow q^{-n}$ and $x \leftarrow q^n x$:

$$\sum_{m\geq 0} (-1)^m q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix} x^m = (x)_n \tag{2}$$

and to the following identity of Euler when $a = 0$:

$$\sum_{m\geq 0} \frac{x^m}{(q)_m} = \frac{1}{(x)_\infty}. \tag{3}$$

Let $n \geq 1$ be a fixed integer and S_n denote the group of permutations of the set $\{1, 2, \ldots, n\}$. Let $X = \{x_1, \ldots, x_n\}$ be a set of indeterminates and q a parameter. For each *partition* $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $\leq n$, if $m_i := m_i(\lambda)$

is the multiplicity of i in λ, then we also note λ by $(1^{m_1}\, 2^{m_2}\, \ldots)$. Recall that the Hall-Littlewood polynomials $P_\lambda(X,q)$ are defined by [7, p.208] :

$$P_\lambda(X,q) = \prod_{i\geq 1} \frac{(1-q)^{m_i}}{(q)_{m_i}} \sum_{w\in S_n} w\left(x_1^{\lambda_1}\ldots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - qx_j}{x_i - x_j}\right),$$

therefore the coefficient of $x_1^{\lambda_1}\ldots x_n^{\lambda_n}$ in P_λ is 1.

Set

$$\Phi(X) := \prod_i \frac{1+qx_i}{1-x_i} \prod_{j<k} \frac{1-q^2 x_j x_k}{1-x_j x_k}.$$

Kawanaka's result reads then as follows [6] :

Theorem 1 (Kawanaka). *We have*

$$\sum_\lambda \left(\prod_{i\geq 1} (-q)_{m_i}\right) P_\lambda(X, q^2) = \Phi(X). \tag{4}$$

For each sequence $\xi \in \{\pm 1\}^n$, set $X^\xi := \{x_1^{\xi_1}, \ldots, x_n^{\xi_n}\}$. Here is our finite extension of Kawanaka's formula.

Theorem 2. *For $k \geq 1$ the following identity holds*

$$\sum_{\lambda_1 \leq k} \left(\prod_{i=1}^{k-1} (-q)_{m_i}\right) P_\lambda(X, q^2) = \sum_{\xi\in\{\pm 1\}^n} \Phi(X^\xi) \prod_i x_i^{k(1-\xi_i)/2}. \tag{5}$$

Remark. In the case $q = 0$, the right-hand side of (5) can be written as a quotient of determinants and the formula reduces to a known identity of Schur functions [4].

For any partition λ it will be convenient to adopt the following notation :

$$(x)_\lambda := (x;\, q)_\lambda = (x)_{\lambda_1-\lambda_2}(x)_{\lambda_2-\lambda_3}\cdots,$$

and to introduce the general q-binomial coefficients

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} := \frac{(q)_n}{(q)_{n-\lambda_1}(q)_\lambda},$$

with the convention that $\begin{bmatrix} n \\ \lambda \end{bmatrix} = 0$ if $\lambda_1 > n$. If $\lambda = (\lambda_1)$ we recover the classical q-binomial coefficient. Finally, for any partition λ we denote by $l(\lambda)$ the length of λ, i.e., the number of its positive parts, and $n(\lambda) := \sum_i \binom{\lambda_i}{2}$. When $x_i = zq^{2i-2}$ for $i \geq 1$ formula (5) specializes to the following identity.

Corollary 1. *For $k \geq 1$ there holds*

$$\sum_{l(\lambda)\leq k} \left(\prod_{i=1}^{k-1} (-q)_{\lambda_i - \lambda_{i+1}} \right) z^{|\lambda|} q^{2n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix}_{q^2} \tag{6}$$
$$= \sum_{r=0}^{n} (-1)^r z^{kr} q^{r+(2k+2)\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \frac{(-z/q)_{2n+1}}{(z^2 q^{2r-2}; q^2)_{n+1}} (1 - z q^{2r-1}).$$

Now, as in [5, 9] and inspired by (6), we can prove the following key q-identity which allows to produce identities of Rogers-Ramanujan type :

Theorem 3. *For $k \geq 1$,*

$$\sum_{l(\lambda)\leq k} z^{|\lambda|} q^{2n(\lambda)} \frac{(a, b; q^{-1})_{\lambda_1}}{(-q)_{\lambda_k} (q)_\lambda} \tag{7}$$
$$= \frac{(-z/q)_\infty}{(abz)_\infty} \sum_{r\geq 0} (-1)^r z^{kr} q^{r+(2k+2)\binom{r}{2}} \frac{(a, b; q^{-1})_r}{(q^2; q^2)_r} \frac{(azq^r, bzq^r)_\infty}{(z^2 q^{2r-2}; q^2)_\infty} (1 - z q^{2r-1}).$$

This paper is organized as follows. In Section 2 we give a new proof of Kawanaka's formula using Pieri's rule for Hall-Littlewood polynomials since Kawanaka's original proof uses the representation theory of groups. In section 3, we derive from Theorem 3 twelve multiple analogs of Rogers-Ramanujan type identities. In section 4 we give the proofs of Theorem 2 and Corollary 1, and some consequences, and defer the elementary proof, i.e., without using the Hall-Littlewood polynomials, of Theorem 3, Corollary 1 and other multiple q-series identities to section 5. To prove Theorems 2 and 3 we apply the generating function technique which was developped in [5, 7, 9].

2 Another proof of Kawanaka's identity

Recall [7, p.230, Ex.1] the following summation of Hall-Littlewood polynomials :

$$\sum_{\mu} P_\mu(X, q) = \prod_i \frac{1}{1 - x_i} \prod_{i<j} \frac{1 - q x_i x_j}{1 - x_i x_j}.$$

By replacing q by q^2, we get

$$\sum_{\mu} P_\mu(X, q^2) = \prod_i \frac{1}{1 - x_i} \prod_{i<j} \frac{1 - q^2 x_i x_j}{1 - x_i x_j}. \tag{8}$$

Note that

$$\sum_{r\geq 0} e_k(X)q^k = \prod_i (1+qx_i), \tag{9}$$

where $e_r(X)$ stands for the r-th elementary symmetric function. Identities (8) and (9) imply

$$\sum_\mu \sum_r q^r P_\mu(X,q^2)e_r(X) = \prod_i \frac{1+qx_i}{1-x_i} \prod_{i<j} \frac{1-q^2x_ix_j}{1-x_ix_j}.$$

From [7, p.209, (2.8)], we have

$$P_{(1^r)}(X,q) = e_r(X),$$

and this shows that

$$\sum_\mu \sum_r q^r P_\mu(X,q^2)P_{(1^r)}(X,q^2) = \prod_i \frac{1+qx_i}{1-x_i} \prod_{i<j} \frac{1-q^2x_ix_j}{1-x_ix_j}.$$

Let $f^\lambda_{\mu\nu}(q)$ be the coefficients defined by

$$P_\mu(X,q)P_\nu(X,q) = \sum_\lambda f^\lambda_{\mu\nu}(q)P_\lambda(X,q),$$

then, by [7, p.215 (3.2)] we have

$$f^\lambda_{\mu(1^m)}(q) = \prod_{i\geq 1} \begin{bmatrix} \lambda'_i - \lambda'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_q$$

(and therefore $f^\lambda_{\mu(1^m)}(q) = 0$ unless $\lambda \setminus \mu$ is a m vertical strip, or m-vs, which means $\lambda \subset \mu$, $|\lambda \setminus \mu| = m$ and there is at most one cell in each row of the Ferrers diagram of $\lambda \setminus \mu$). Thus we have

$$\sum_\lambda \sum_{\substack{\mu \\ \lambda\setminus\mu \text{ vs}}} q^{|\lambda-\mu|} \prod_{i\geq 1} \begin{bmatrix} \lambda'_i - \lambda'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_{q^2} P_\lambda(X,q^2)$$
$$= \prod_i \frac{1+qx_i}{1-x_i} \prod_{i<j} \frac{1-q^2x_ix_j}{1-x_ix_j}.$$

Applying the identity (see [1], and [10] for a bijective proof) :

$$\sum_{k=0}^{n} q^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} = \prod_{k=1}^{n}(1+q^k), \tag{10}$$

we conclude that

$$\sum_{\substack{\mu \\ \lambda\backslash\mu \text{ vs}}} q^{|\lambda-\mu|} \prod_{i\geq 1} \begin{bmatrix} \lambda_i'-\lambda_{i+1}' \\ \lambda_i'-\mu_i' \end{bmatrix}_{q^2} = \prod_{i\geq 1} \prod_{k=1}^{\lambda_i'-\lambda_{i+1}'} (1+q^k),$$

which is precisely what we desired to prove.
Remark. Kawanaka [6, (5.2)] proved another identity for Hall-Littlewood polynomials :

$$\sum_{\lambda} q^{o(\lambda)/2} \left(\prod_{v\in\lambda}(1-q^{l(v)+1}) \right) P_\lambda(X,q) = \prod_{i\leq j} \frac{1-qx_ix_j}{1-x_ix_j}, \tag{11}$$

where the sum on the left is taken over all partitions λ such that $m_i(\lambda)$ is even for odd i, and

$$o(\lambda) = \sum_{i \text{ odd}} m_i(\lambda).$$

It would be possible to prove this identity in the same manner as above.

There is a related identity about Hall-Littlewood polynomials in Macdonald's book [7, p. 219] :

$$\sum_{\lambda} q^{n(\lambda)} \left(\prod_{j=1}^{l(\lambda)}(1+q^{1-j}y) \right) P_\lambda(X,q) = \prod_{i\geq 1} \frac{1+x_iy}{1-x_i}. \tag{12}$$

3 Multiple identities of Rogers-Ramanujan type

For any partition λ set $n_2(\lambda) = \sum_i \lambda_i^2$. We shall derive several identities of Rogers-Ramanujan type from Theorem 3.

First we note that if $z=q^2$ identity (7) reduces to

$$\sum_{l(\lambda)\le k} q^{|\lambda|+n_2(\lambda)}\frac{(a,b;q^{-1})_{\lambda_1}}{(-q)_{\lambda_k}(q)_\lambda} = \frac{1}{(q,abq^2)_\infty} \tag{13}$$
$$\times \sum_{r\ge 0}(-1)^r q^{(2k+1)r+(2k+2)\binom{r}{2}}(a,b;q^{-1})_r(aq^{r+2},bq^{r+2})_\infty(1-zq^{2r-1}),$$

and if $z=q$ it becomes

$$\sum_{l(\lambda)\le k} q^{n_2(\lambda)}\frac{(a,b;q^{-1})_{\lambda_1}}{(-q)_{\lambda_k}(q)_\lambda} = \frac{1}{(q,abq)_\infty} \tag{14}$$
$$\times\left((aq,bq)_\infty + 2\sum_{r\ge 1}(-1)^r q^{(k+1)r^2/2}(a,b;q^{-1})_r(aq^{r+1},bq^{r+1})_\infty\right).$$

We need the two following forms of *Jacobi's triple product* identity [1, p.21] :

$$J(x,\,q) := (q,\,x,\,q/x)_\infty \;=\; \sum_{r=0}^{\infty}(-1)^r x^r q^{\binom{r}{2}}(1-q^{2r+1}/x^{2r+1}) \tag{15}$$
$$=\; 1+\sum_{r=1}^{\infty}(-1)^r x^r q^{\binom{r}{2}}(1+q^r/x^{2r}). \tag{16}$$

Theorem 4. *For $k\ge 1$, the following identities hold*

$$\sum_{l(\lambda)\le k}\frac{q^{|\lambda|+n_2(\lambda)}}{(-q)_{\lambda_k}(q)_\lambda} = \frac{(q^{2k+2},q^{2k+1},q;q^{2k+2})_\infty}{(q)_\infty}, \tag{17}$$
$$\sum_{l(\lambda)\le k}\frac{q^{|\lambda|+n_2(\lambda)-(\lambda_1^2+\lambda_1)/2}(-q)_{\lambda_1}}{(-q)_{\lambda_k}(q)_\lambda} = \frac{(-q)_\infty}{(q)_\infty}(q^{2k+1},q^{2k},q;q^{2k+1})_\infty, \tag{18}$$
$$\sum_{l(\lambda)\le k}\frac{q^{2|\lambda|+2n_2(\lambda)-\lambda_1^2}(-q;q^2)_{\lambda_1}}{(-q^2;q^2)_{\lambda_k}(q^2;q^2)_\lambda} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^{4k+2},q^{4k+1},q;q^{4k+2})_\infty, \tag{19}$$
$$\sum_{l(\lambda)\le k}\frac{q^{2|\lambda|+2n_2(\lambda)-2\lambda_1^2-\lambda_1}(-q)_{2\lambda_1}}{(-q^2;q^2)_{\lambda_k}(q^2;q^2)_\lambda} = \frac{(-q)_\infty}{(q)_\infty}(q^{4k},q^{4k-1},q;q^{4k})_\infty, \tag{20}$$

$$\sum_{l(\lambda)\le k}\frac{q^{n_2(\lambda)}}{(-q)_{\lambda_k}(q)_\lambda} = \frac{(q^{2k+2},q^{k+1},q^{k+1};q^{2k+2})_\infty}{(q)_\infty}, \quad (21)$$

$$\sum_{l(\lambda)\le k}\frac{q^{n_2(\lambda)-(\lambda_1^2+\lambda_1)/2}(-q)_{\lambda_1}}{(-q)_{\lambda_k}(q)_\lambda} = \frac{(-1)_\infty}{(q)_\infty}(q^{2k+1},q^{k},q^{k+1};q^{2k+1})_\infty, \quad (22)$$

$$\sum_{l(\lambda)\le k}\frac{q^{2n_2(\lambda)-\lambda_1^2}(-q;q^2)_{\lambda_1}}{(-q^2;q^2)_{\lambda_k}(q^2;q^2)_\lambda} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^{4k+2},q^{2k+1},q^{2k+1};q^{4k+2})_\infty. \quad (23)$$

Proof. For (17)-(20), set respectively $(a,b)=(0,0)$, $(-q^{-1},0)$, $(-q^{-1/2},0)$ and $(-q^{-1/2},-q^{-1})$ in (13), and then apply (15).

For (21)-(22), set respectively $(a,b)=(0,0)$, $(-q^{-1/2},0)$ and $(-q^{-1},0)$ in (14), and then apply (16). Moreover, for (19), (20) and (23), replace at last q by q^2. □

Theorem 5. *For $k\ge 1$, the following identities hold*

$$\sum_{l(\lambda)\le k}\frac{q^{|\lambda|+n_2(\lambda)-(\lambda_1^2+3\lambda_1)/2}(-q)_{\lambda_1}(1-q^{\lambda_1})}{(-q)_{\lambda_k}(q)_\lambda} = \frac{(-q)_\infty}{(q)_\infty}(q^{2k+1},q^{2k-1},q^{2};q^{2k+1})_\infty, \quad (24)$$

$$\sum_{l(\lambda)\le k}\frac{q^{2|\lambda|+2n_2(\lambda)-\lambda_1^2-2\lambda_1}(-q;q^2)_{\lambda_1}}{(-q^2;q^2)_{\lambda_k}(q^2;q^2)_\lambda} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^{4k+2},q^{4k-1},q^{3};q^{4k+2})_\infty, \quad (25)$$

$$\sum_{l(\lambda)\le k}\frac{q^{|\lambda|+n_2(\lambda)-\lambda_1}}{(-q)_{\lambda_k}(q)_\lambda} = \frac{(q^{2k+2},q^{2k},q^{2};q^{2k+2})_\infty}{(q)_\infty}, \quad (26)$$

$$\sum_{l(\lambda)\le k}\frac{q^{|\lambda|+n_2(\lambda)-2\lambda_1}(1-q^{2\lambda_1})}{(-q)_{\lambda_k}(q)_\lambda} = \frac{(q^{2k+2},q^{2k-1},q^{3};q^{2k+2})_\infty}{(q)_\infty}, \quad (27)$$

$$\sum_{l(\lambda)\le k}\frac{q^{n_2(\lambda)-\lambda_1}}{(-q)_{\lambda_k}(q)_\lambda} = \frac{(-1)_\infty}{(q^2;q^2)_\infty}(q^{2k+2},q^{k},q^{k+2};q^{2k+2})_\infty. \quad (28)$$

Proof. For $i \in \{0,1,2\}$, denote by $[b^i]$ the operation of extracting the coefficient of b^i in the corresponding identity. For (24)-(27), apply the following operations to (13) respectively : $a=-q^{-1}$ and $(1-1/q)[b]$, $a=0$ and $[b^0]+(1-1/q)[b]$, $a=-q^{-1/2}$ and $[b^0]+(1-1/q)[b]$, $a=0$ and $[b]+(1-1/q)[b^2]$, and then apply (15). Moreover, for (25), replace at last q by q^2.

For (28) apply the operations $a=0$ and $(1-1/q)[b]$ to (14) and then apply (16). $\square$

When $k=1$, identities (17), (18), (20), (22) and (27) reduce to special cases of the q-binomial identity (1). For example, when $k=1$ identity (20) reduces to

$$\sum_{n=0}^{\infty}\frac{q^n(-q;q^2)_n}{(q^2;q^2)_n}=\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty},$$

which is the q-binomial identity (1) after substitutions $q \leftarrow q^2$, $a \leftarrow -q$ and $x \leftarrow q$. The other identities reduce when $k=1$ to the following Rogers-Ramanujan type identities :

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q,q^5,q^6;q^6)_\infty, \tag{29}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^2;q^2)_n} = \frac{(q^2,q^2,q^4;q^4)_\infty}{(q)_\infty}, \tag{30}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^3,q^3,q^6;q^6)_\infty. \tag{31}$$

Note that (30) and (31) are identities (23) and (25) in Slater's list [8].

4 Proof of Theorem 2 and consequences

4.1 Proof of Theorem 2

For any statement A it will be convenient to use the true or false function $\chi(A)$, which is 1 if A is true and 0 if A is false. Consider the generating function

$$S(u)=\sum_{\lambda_0,\lambda}\left(\prod_{i=1}^{\lambda_0-1}(-q)_{m_i}\right)P_\lambda(X,q^2)\,u^{\lambda_0}$$

where the sum is over all partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and the integers $\lambda_0 \geq \lambda_1$. Suppose $\lambda = (\mu_1^{r_1}\, \mu_2^{r_2}\, \ldots \mu_k^{r_k})$, where $\mu_1 > \mu_2 > \cdots > \mu_k \geq 0$ and $(r_1, \ldots, r_k)$ is a composition of n.

Let S_n^λ be the set of permutations of S_n which fix λ. Each $w \in S_n/S_n^\lambda$ corresponds to a surjective mapping $f : X \longrightarrow \{1, 2, \ldots, k\}$ such that $|f^{-1}(i)| = r_i$. For any subset Y of X, let $p(Y)$ denote the product of the elements of Y (in particular, $p(\emptyset) = 1$). We can rewrite Hall-Littlewood functions as follows :

$$P_\lambda(X, q^2) = \sum_f p(f^{-1}(1))^{\mu_1} \cdots p(f^{-1}(k))^{\mu_k} \prod_{f(x_i)<f(x_j)} \frac{x_i - q^2 x_j}{x_i - x_j},$$

summed over all surjective mappings $f : X \longrightarrow \{1, 2, \ldots, k\}$ such that $|f^{-1}(i)| = r_i$. Furthermore, each such f determines a *filtration* of X :

$$\mathcal{F} : \quad \emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = X, \tag{32}$$

according to the rule $x_i \in F_l \iff f(x_i) \leq l$ for $1 \leq l \leq k$. Conversely, such a filtration $\mathcal{F} = (F_0, F_1, \ldots, F_k)$ determines a surjection $f : X \longrightarrow \{1, 2, \ldots, k\}$ uniquely. Thus we can write :

$$P_\lambda(X, q^2) = \sum_{\mathcal{F}} \pi_{\mathcal{F}} \prod_{1\leq i\leq k} p(F_i \setminus F_{i-1})^{\mu_i}, \tag{33}$$

summed over all the filtrations $\mathcal{F}$ such that $|F_i| = r_1 + r_2 + \cdots + r_i$ for $1 \leq i \leq k$, and

$$\pi_{\mathcal{F}} = \prod_{f(x_i)<f(x_j)} \frac{x_i - q^2 x_j}{x_i - x_j},$$

where f is the function defined by $\mathcal{F}$.

Now let $\nu_i = \mu_i - \mu_{i+1}$ if $1 \leq i \leq k-1$ and $\nu_k = \mu_k$, thus $\nu_i > 0$ if $i < k$ and $\nu_k \geq 0$. Furthermore, let $\mu_0 = \lambda_0$ and $\nu_0 = \mu_0 - \mu_1$ in the definition of $S(u)$, so that $\nu_0 \geq 0$ and $\mu_0 = \nu_0 + \nu_1 + \cdots + \nu_k$. Define $c_{\mathcal{F}} = \prod_{i=1}^k (-q)_{|F_i \setminus F_{i-1}|}$ for any filtration $\mathcal{F}$. Thus, since the lengths of columns of λ are $|F_j| = r_1 + \cdots + r_j$ with multiplicities ν_j and $r_j = m_{\mu_j}(\lambda)$ for $1 \leq j \leq k$, we have

$$\begin{aligned} \prod_{i=1}^{\lambda_0 - 1} (-q)_{m_i} &= c_{\mathcal{F}} \times \left(\chi(\nu_k = 0)(-q)_{|F_k \setminus F_{k-1}|} + \chi(\nu_k \neq 0)\right)^{-1} \\ &\qquad \times \left(\chi(\nu_0 = 0)(-q)_{|F_1|} + \chi(\nu_0 \neq 0)\right)^{-1}. \end{aligned}$$

Let $F(X)$ be the set of filtrations of X. Summarizing we obtain

$$\begin{aligned} S(u) &= \sum_{\mathcal{F}\in F(X)} c_{\mathcal{F}}\, \pi_{\mathcal{F}} \sum_{\nu_1>0} (u\, p(F_1))^{\nu_1} \cdots \sum_{\nu_{k-1}>0} (u\, p(F_{k-1}))^{\nu_{k-1}} \\ &\qquad \times \sum_{\nu_0\geq 0} \frac{u^{\nu_0}}{\chi(\nu_0=0)\,(-q)_{|F_1|} + \chi(\nu_0\neq 0)} \\ &\qquad \times \sum_{\nu_k\geq 0} \frac{u^{\nu_k}\, p(F_k)^{\nu_k}}{\chi(\nu_k=0)\,(-q)_{|F_k\setminus F_{k-1}|} + \chi(\nu_k\neq 0)}. \end{aligned} \tag{34}$$

For any filtration $\mathcal{F}$ of X set

$$\mathcal{A}_{\mathcal{F}}(X,u) = c_{\mathcal{F}} \prod_j \left[\frac{p(F_j)u}{1-p(F_j)u} + \frac{\chi(F_j=X)}{(-q)_{|F_j\setminus F_{j-1}|}} + \frac{\chi(F_j=\emptyset)}{(-q)_{|F_1|}} \right].$$

It follows from (34) that

$$S(u) = \sum_{\mathcal{F}\in F(X)} \pi_{\mathcal{F}} \mathcal{A}_{\mathcal{F}}(X,u).$$

Hence $S(u)$ is a rational function of u with simple poles at $1/p(Y)$, where Y is a subset of X. We are now proceeding to compute the corresponding residue $c(Y)$ at each pole $u=1/p(Y)$.

Let us start with $c(\emptyset)$. Writing $\lambda_0=\lambda_1+k$ with $k\geq 0$, we see that

$$\begin{aligned} S(u) &= \sum_{\lambda} f_\lambda(q) P_\lambda(X,q^2) u^{\lambda_1} \sum_{k\geq 0} \frac{u^k}{\chi(k=0)(-q)_{m_{\lambda_1}} + \chi(k\neq 0)} \\ &= \sum_{\lambda} f_\lambda(q) P_\lambda(X,q^2) u^{\lambda_1} \left(\frac{u}{1-u} + \frac{1}{(-q)_{m_{\lambda_1}}} \right). \end{aligned}$$

It follows from (4) that

$$c(\emptyset) = [S(u)(1-u)]_{u=1} = \Phi(X).$$

For the computations of other residues, we need some more notations. For any $Y\subseteq X$, let $Y'=X\setminus Y$ and $-Y=\{x_i^{-1} : x_i\in Y\}$. Then

$$c(Y) = \left[\sum_{\mathcal{F}} \pi_{\mathcal{F}} \mathcal{A}_{\mathcal{F}}(X,\, u)(1-p(Y)u) \right]_{u=p(-Y)}. \tag{35}$$

If $Y \notin \mathcal{F}$, the corresponding summand is equal to 0. Thus we need only to consider the following filtrations $\mathcal{F}$:

$$\emptyset = F_0 \subsetneq \cdots \subsetneq F_t = Y \subsetneq \cdots \subsetneq F_k = X \qquad 1 \leq t \leq k.$$

We may then split $\mathcal{F}$ into two filtrations $\mathcal{F}_1$ and $\mathcal{F}_2$:

$$\begin{aligned} \mathcal{F}_1 &: \emptyset \subsetneq -(Y \setminus F_{t-1}) \subsetneq \cdots \subsetneq -(Y \setminus F_1) \subsetneq -Y, \\ \mathcal{F}_2 &: \emptyset \subsetneq F_{t+1} \setminus Y \subsetneq \cdots \subsetneq F_{k-1} \setminus Y \subsetneq Y'. \end{aligned}$$

Then, writing $v = p(Y)u$ and $c_{\mathcal{F}} = c_{\mathcal{F}_1} \times c_{\mathcal{F}_2}$, we have

$$\pi_{\mathcal{F}}(X) = \pi_{\mathcal{F}_1}(-Y)\pi_{\mathcal{F}_2}(Y') \prod_{x_i \in Y, x_j \in Y'} \frac{1 - q^2 x_i^{-1} x_j}{1 - x_i^{-1} x_j},$$

and $\mathcal{A}_{\mathcal{F}}(X,\, u)(1 - p(Y)u)$ is equal to

$$\begin{aligned} &\mathcal{A}_{\mathcal{F}_1}(-Y,\, v)\mathcal{A}_{\mathcal{F}_2}(Y',\, v)(1 - v) \left(\frac{v}{1 - v} + \frac{\chi(Y = X)}{(-q)_{|Y \setminus F_{t-1}|}} \right) \\ &\times \left(\frac{v}{1 - v} + \frac{1}{(-q)_{|Y \setminus F_{t-1}|}} \right)^{-1} \left(\frac{v}{1 - v} + \frac{1}{(-q)_{|F_{t+1} \setminus Y|}} \right)^{-1}. \end{aligned}$$

Thus when $u = p(-Y)$, i.e., $v = 1$,

$$\begin{aligned} &[\pi_{\mathcal{F}}(X)\mathcal{A}_{\mathcal{F}}(X,\, u)(1 - p(Y)u)]_{u=p(-Y)} = \\ &\qquad [\pi_{\mathcal{F}_1}(-Y)\mathcal{A}_{\mathcal{F}_1}(-Y,\, v)(1 - v)\pi_{\mathcal{F}_2}(Y')\mathcal{A}_{\mathcal{F}_2}(Y',\, v)(1 - v)]_{v=1} \\ &\qquad\qquad \times \prod_{x_i \in Y, x_j \in Y'} \frac{1 - q^2 x_i^{-1} x_j}{1 - x_i^{-1} x_j}. \end{aligned}$$

Using (35) and the result of $c(\emptyset)$, which can be written

$$\left[\sum_{\mathcal{F}} \pi_{\mathcal{F}} \mathcal{A}_{\mathcal{F}}(X, u)(1 - u) \right]_{u=1} = \Phi(X),$$

we get

$$c(Y) = \Phi(-Y)\Phi(Y') \prod_{x_i \in Y, x_j \in Y'} \frac{1 - q^2 x_i^{-1} x_j}{1 - x_i^{-1} x_j}.$$

Each subset Y of X can be encoded by a sequence $\xi \in \{\pm 1\}^n$ according to the rule : $\xi_i = 1$ if $x_i \notin Y$ and $\xi_i = -1$ if $x_i \in Y$. Hence

$$c(Y) = \Phi(X^\xi).$$

Note also that

$$p(Y) = \prod_i x_i^{(1-\xi_i)/2}, \qquad p(-Y) = \prod_i x_i^{(\xi_i-1)/2}.$$

Now, extracting the coefficients of u^k in the equation :

$$S(u) = \sum_{Y \subseteq X} \frac{c(Y)}{1 - p(Y)u},$$

yields

$$\sum_{\lambda_1 \le k} \left(\prod_{i=1}^{k-1} (-q)_{m_i} \right) P_\lambda(X, q^2) = \sum_{Y \subseteq X} c(Y) p(Y)^k.$$

Finally, substituting the value of $c(Y)$ in the above formula we obtain (5).

4.2 Some direct consequences on q-series

We recall and prove identity (6) in Corollary 1, which was the crucial souce of inspiration for finding identity (7) in Theorem 3 :

$$\sum_{l(\lambda) \le k} \left(\prod_{i=1}^{k-1} (-q)_{\lambda_i - \lambda_{i+1}} \right) z^{|\lambda|} q^{2n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix}_{q^2}$$
$$= \sum_{r=0}^{n} (-1)^r z^{kr} q^{r+(2k+2)\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \frac{(-z/q)_{2n+1}}{(z^2 q^{2r-2}; q^2)_{n+1}} (1 - z q^{2r-1}).$$

Proof. We know [7, p. 213] that if $x_i = zq^{2i-2}$ $(1 \le i \le n)$ then :

$$P_{\lambda'}(X, q^2) = z^{|\lambda|} q^{2n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix}_{q^2}. \tag{36}$$

Taking the conjugation in the left-hand side of (5) we obtain the left-hand side of (6). Set

$$\Psi(X) = \prod_i \frac{1}{1 - x_i^2} \prod_{j<k} \frac{1 - q^2 x_j x_k}{1 - x_j x_k}.$$

Then, for any $\xi \in \{\pm 1\}^n$ such that the number of $\xi_i = -1$ is r, $0 \le r \le n$, we can write $\Phi(X^\xi)$ as follows:

$$\Phi(X^\xi) = \Psi(X^\xi) \prod_i \frac{1 + q x_i^{\xi_i}}{1 - x_i^{\xi_i}} (1 - x_i^{2\xi_i}), \tag{37}$$

which is readily seen to equal 0 unless $\xi \in \{-1\}^r \times \{1\}^{n-r}$. Now, in the latter case, we have $\prod_i x_i^{k(1-\xi_i)/2} = z^{kr} q^{2k\binom{r}{2}}$,

$$\prod_{i=1}^{n} \frac{1 + q x_i^{\xi_i}}{1 - x_i^{\xi_i}} (1 - x_i^{2\xi_i}) = \frac{(z^2; q^4)_n}{z^{2r} q^{4\binom{r}{2} - r}} \frac{(-z/q; q^2)_r}{(z; q^2)_r} \frac{(-z q^{2r+1}; q^2)_{n-r}}{(z q^{2r}; q^2)_{n-r}}, \tag{38}$$

and [9, p. 476] :

$$\Psi(X^\xi) = (-1)^r z^{2r} q^{6\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix} \frac{1 - z^2 q^{4r-2}}{(z q^{r-1})_{n+1}}. \tag{39}$$

Substituting these into the right side of (5) we obtain the right side of (6) after simple manipulations. $\square$

When $n \to +\infty$, since $\begin{bmatrix} n \\ \lambda \end{bmatrix} \to \frac{1}{(q)_\lambda}$, equation (6) reduces to :

$$\sum_{l(\lambda) \le k} \frac{z^{|\lambda|} q^{2n(\lambda)}}{(-q)_{\lambda_k} (q)_\lambda} = (-z/q)_\infty \sum_{r \ge 0} \frac{(-1)^r z^{kr} q^{r + (2k+2)\binom{r}{2}}}{(q^2; q^2)_r (z^2 q^{2r-2})_\infty} (1 - z q^{2r-1}). \tag{40}$$

Furthermore, as it was done in section 2, setting $z = q^2$ and $z = q$ in (40) yields respectively (17) and (21).

5 Elementary approach and proof of Theorem 3

5.1 Preliminaries

We will need the following result, which corresponds to the case $k \to \infty$ in (6), and can be proved in an elementary way :

Lemma 1. *For $n \geq 0$*

$$\sum_{\lambda} z^{|\lambda|} q^{2n(\lambda)} (-q)_\lambda \begin{bmatrix} n \\ \lambda \end{bmatrix}_{q^2} = \frac{(-z)_{2n}}{(z^2;q^2)_n}. \tag{41}$$

Proof. Recall the following identity, which is proved in [5] :

$$q^{\binom{m}{2}+n(\mu)} \begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} n \\ \mu \end{bmatrix} = \sum_{\lambda} q^{n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix} \prod_{i\geq 1} \begin{bmatrix} \lambda_i - \lambda_{i+1} \\ \lambda_i - \mu_i \end{bmatrix}, \tag{42}$$

where the sum is over all partitions λ such that λ/μ is an m-horizontal strip, i.e., $\mu \subseteq \lambda$, $|\lambda/\mu| = m$ and there is at most one cell in each column of the Ferrers diagram of λ/μ.
We also need

$$\sum_{\lambda} z^{|\lambda|} q^{n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix} = \frac{(-z)_n}{(z^2)_n}, \tag{43}$$

which can be found in [5, 9].
Using (43) with q replaced by q^2 and (2), the right-hand side of (41) can be written

$$\begin{aligned} \frac{(-z;q^2)_n}{(z^2;q^2)_n}(-zq;q^2)_n &= \sum_{\mu,\, m} z^{|\mu|} q^{2n(\mu)} \begin{bmatrix} n \\ \mu \end{bmatrix}_{q^2} z^m q^{2\binom{m}{2}+m} \begin{bmatrix} n \\ m \end{bmatrix}_{q^2} \\ &= \sum_{\lambda,\, m} z^{|\lambda|} q^{2n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix}_{q^2} \prod_{i\geq 1} \sum_{r_i \geq 0} q^{r_i} \begin{bmatrix} \lambda_i - \lambda_{i+1} \\ r_i \end{bmatrix}_{q^2}, \end{aligned}$$

where the last equality follows from (42), setting $r_i = \lambda_i - \mu_i$ for $i \geq 1$. Now we conclude by using (10). $\square$

Recall the following extension of the $n \to \infty$ case of (41), which is Stembridge's lemma 3.3 (b) in [9], and identity (60) in [5] :

$$\sum_{\lambda} z^{|\lambda|} q^{2n(\lambda)} \frac{(a,b;q^{-1})_{\lambda_1}}{(q)_\lambda} = \frac{(az,bz)_\infty}{(z,abz)_\infty}. \tag{44}$$

Now, using (41), we are able to prove directly identity (6) in Corollary 1, and then using (44), to deduce an elementary proof of (7) in Theorem 3.

5.2 Elementary proof of Corollary 1

Consider the generating function of the left-hand side of (6) :

$$\begin{aligned}
\varphi(u) &= \sum_{k\geq 0} u^k \sum_{l(\lambda)\leq k} \frac{(-q)_\lambda}{(-q)_{\lambda_k}} z^{|\lambda|} q^{2n(\lambda)} \begin{bmatrix} n \\ \lambda \end{bmatrix}_{q^2} && (45)\\
&= \sum_{\lambda} u^{l(\lambda)} z^{|\lambda|} q^{2n(\lambda)} (-q)_\lambda \begin{bmatrix} n \\ \lambda \end{bmatrix}_{q^2} \sum_{k\geq 0} \frac{u^k}{(-q)_{\lambda_{k+l(\lambda)}}} \\
&= \sum_{\lambda} u^{l(\lambda)} z^{|\lambda|} q^{2n(\lambda)} (-q)_\lambda \begin{bmatrix} n \\ \lambda \end{bmatrix}_{q^2} \left(\frac{u}{1-u} + \frac{1}{(-q)_{\lambda_{l(\lambda)}}} \right). && (46)
\end{aligned}$$

Now, each partition λ with parts bounded by n can be encoded by a pair of sequences $\nu = (\nu_0, \nu_1, \cdots, \nu_l)$ and $\mathbf{m} = (m_0, \cdots, m_l)$ such that $\lambda = (\nu_0^{m_0}, \ldots, \nu_l^{m_l})$, where $n = \nu_0 > \nu_1 > \cdots > \nu_l > 0$ and ν_i has multiplicity $m_i \geq 1$ for $1 \leq i \leq l$ and $\nu_0 = n$ has multiplicity $m_0 \geq 0$. Using the notation :

$$< \alpha >= \frac{\alpha}{1-\alpha}, \quad u_i = z^i q^{i(i-1)} \quad \text{for} \quad i \geq 0,$$

we can then rewrite (46) as follows :

$$\begin{aligned}
\varphi(u) &= \sum_{\nu} (-q)_\nu \begin{bmatrix} n \\ \nu \end{bmatrix}_{q^2} \left(< u > + \frac{1}{(-q)_{\nu_l}} \right) \\
&\quad \times \sum_{\mathbf{m}} \left((u_n u)^{m_0} + \frac{\chi(m_0 = 0)}{(-q)_{n-\nu_1}} \right) \prod_{i=1}^{l} (u_{\nu_i} u)^{m_i} \\
&= \sum_{\nu} \frac{(q^2;q^2)_n}{(q)_\nu} B_\nu, && (47)
\end{aligned}$$

where the sum is over all strict partitions $\nu = (\nu_0, \nu_1, \ldots, \nu_l)$ and

$$B_\nu = \left(< u > + \frac{1}{(-q)_{\nu_l}} \right) \left(< u_r u > + \frac{1}{(-q)_{n-\nu_1}} \right) \prod_{i=1}^{l} < u_{\nu_i} u > .$$

So $\varphi(u)$ is a rational fraction with simple poles at u_r^{-1} for $0 \leq r \leq n$. Let $b_r(z,n)$ be the corresponding residue of $\varphi(u)$ at u_r^{-1} for $0 \leq r \leq n$. Then, it follows from (47) that

$$b_r(z,n) = \sum_{\nu} \frac{(q^2;q^2)_n}{(q)_\nu} \left[B_\nu (1 - u_r u) \right]_{u=u_r^{-1}} . \qquad (48)$$

We shall first consider the cases where $r=0$ or n. Using (46) and (41) we have

$$b_0(z,n)=[\varphi(u)(1-u)]_{u=1}=\frac{(-z)_{2n}}{(z^2;q^2)_n}. \tag{49}$$

Now, by (47) and(48) we have

$$b_0(z,n)=\sum_\nu \frac{(q^2;q^2)_n}{(q)_\nu}\left(< u_n > +\frac{1}{(-q)_{n-\nu_1}}\right)\prod_{i=1}^{l} < u_{\nu_i} >, \tag{50}$$

and

$$b_n(z,n)=\sum_\nu \frac{(q^2;q^2)_n}{(q)_\nu}\left(< 1/u_n > +\frac{1}{(-q)_{\nu_l}}\right)\prod_{i=1}^{l} < u_{\nu_i}/u_n >, \tag{51}$$

which, by setting $\mu_i=n-\nu_{l+1-i}$ for $1\le i\le l$ and $\mu_0=n$, can be written as

$$b_n(z,n)=\sum_\mu \frac{(q^2;q^2)_n}{(q)_\mu}\left(< 1/u_n > +\frac{1}{(-q)_{n-\mu_1}}\right)\prod_{i=1}^{l} < u_{n-\mu_i}/u_n > . \tag{52}$$

Comparing (52) with (50) we see that $b_n(z,n)$ is equal to $b_0(z,n)$ with z replaced by $z^{-1}q^{-2n+2}$. Il follows from (49) that

$$b_n(z,n)=b_0(z^{-1}q^{-2n+2},n)=(-1)^nq^{n^2}\frac{(-z/q)_{2n}}{(z^2q^{2n-2};q^2)_n}. \tag{53}$$

Consider now the case where $0<r<n$. Clearly, for each partition ν, the corresponding summand in (48) is not zero only if $\nu_j=r$ for some j, $0\le j\le n$. Furthermore, each such partition ν can be splitted into two strict partitions $\rho=(\rho_0,\rho_1,\ldots,\rho_{j-1})$ and $\sigma=(\sigma_0,\ldots,\sigma_{l-j})$ such that $\rho_i=\nu_i-r$ for $0\le i\le j-1$ and $\sigma_s=\nu_{j+s}$ for $0\le s\le l-j$. So we can write (48) as follows :

$$b_r(z,n) \quad = \quad \begin{bmatrix} n \\ r \end{bmatrix}_{q^2}\sum_\rho \frac{(q^2;q^2)_{n-r}}{(q)_\rho}F_\rho(r)\times\sum_\sigma\frac{(q^2;q^2)_r}{(q)_\sigma}G_\sigma(r)$$

where for $\rho=(\rho_0,\rho_1,\ldots,\rho_l)$ with $\rho_0=n-r$,

$$F_\rho(r)=\left(< u_n/u_r > +\frac{1}{(-q)_{n-r-\rho_1}}\right)\prod_{i=1}^{l(\rho)} < u_{\rho_i+r}/u_r >,$$

and for $\sigma = (\sigma_0, \ldots, \sigma_l)$ with $\sigma_0 = n$,

$$G_\sigma(r) = \left(< 1/u_r > + \frac{1}{(-q)_{\sigma_l}}\right) \prod_{i=1}^{l(\sigma)} < u_{\sigma_i}/u_r > .$$

Comparing with (50) and (52) and using (49) and (53) we obtain

$$\begin{aligned} b_r(z,n) &= \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} b_0(zq^{2r}, n-r)\, b_r(z,r) \\ &= (-1)^r q^{r+2\binom{n}{r}} \frac{(-z/q)_{2n+1}}{(z^2q^{2r-2}, q^2)_{n+1}} (1 - zq^{4r-1}). \end{aligned}$$

Finally, extracting the coefficients of u^k in the equation

$$\varphi(u) = \sum_{p=0}^{n} \frac{b_r(z,n)}{1 - u_r u},$$

and using the values for $b_r(z,n)$ we obtain (6).

5.3 Proof of Theorem 3

Consider the generating function of the left-hand side of (7) :

$$\begin{aligned} \varphi_{ab}(u) &:= \sum_{k\geq 0} u^k \sum_{l(\lambda)\leq k} z^{|\lambda|} q^{2n(\lambda)} \frac{(a,\, b; q^{-1})_{\lambda_1}}{(q)_\lambda (-q)_{\lambda_k}} \\ &= \sum_{\lambda} \sum_{k\geq 0} u^{k+l(\lambda)} z^{|\lambda|} q^{2n(\lambda)} \frac{(a,\, b; q^{-1})_{\lambda_1}}{(q)_\lambda (-q)_{\lambda_{l(\lambda)+k}}} \\ &= \sum_{\lambda} u^{l(\lambda)} z^{|\lambda|} q^{2n(\lambda)} \frac{(a,\, b; q^{-1})_{\lambda_1}}{(q)_\lambda} \left(\frac{u}{1-u} + \frac{1}{(-q)_{\lambda_{l(\lambda)}}}\right), \qquad (54) \end{aligned}$$

where the sum is over all the partitions λ. As in the elementary proof of Corollary 1, we can replace any partition λ by a pair $(\nu, \mathbf{m})$, where ν is a strict partition consisting of distinct parts $\nu_1, \cdots, \nu_l$ of λ, so that $\nu_1 > \cdots > \nu_l > 0$, and $\mathbf{m} = (m_1, \ldots, m_l)$ is the sequence of multiplicities of ν_i for $1 \leq i \leq l$. Therefore

$$\begin{aligned} \varphi_{ab}(u) &= \sum_{\nu, \mathbf{m}} \frac{(a,\, b; q^{-1})_{\nu_1}}{(q)_\nu} \left(\frac{u}{1-u} + \frac{1}{(-q)_{\nu_l}}\right) \prod_{i=1}^{l} (u_{\nu_i} u)^{m_i} \\ &= \sum_{\nu} \frac{(a,\, b; q^{-1})_{\nu_1}}{(q)_\nu} \left(< u > + \frac{1}{(-q)_{\nu_l}}\right) \prod_{i=1}^{l} < u_{\nu_i} u >, \qquad (55) \end{aligned}$$

where the sum is over all the strict partitions ν. Each of the terms in this sum, as a rational function of u, has a finite set of simple poles, which may occur at the points u_r^{-1} for $r \geq 0$. Therefore, each term is a linear combination of partial fractions. Moreover, the sum of their expansions converges coefficientwise. So φ_{ab} has an expansion

$$\varphi_{ab}(u) = \sum_{r\geq 0} \frac{c_r}{1 - uz^r q^{r(r-1)}},$$

where c_r denotes the formal sum of partial fraction coefficients contributed by the terms of (55). It remains to compute these residues c_r ($r \geq 0$). By using (44) and (54), we get immediately

$$c_0 = [\varphi_{ab}(u)(1-u)]_{u=1} = \frac{(az,\, bz)_\infty}{(z,\, abz)_\infty}.$$

In view of (55), this yields the identity

$$\sum_{\nu} \frac{(a,\, b; q^{-1})_{\nu_1}}{(q)_\nu} \prod_{i=1}^{l} < u_{\nu_i} >= \frac{(az,\, bz)_\infty}{(z, abz)_\infty}. \tag{56}$$

Clearly, a summand in (55) has a non zero contribution to c_r ($r > 0$) only if the corresponding partition ν has a part equal to r. For any partition ν such that $\exists j \,|\, \nu_j = r$, set $\rho_i := \nu_i - r$ for $1 \leq i < j$ and $\sigma_i := \nu_{i+j}$ for $0 \leq i \leq l-j$, we then get two partitions ρ and σ, with σ_i bounded by r. Multiplying (55) by $(1 - u_r u)$ and setting $u = 1/u_r$ we obtain

$$\begin{aligned} c_r \;&=\; \sum_{\rho} \frac{(a, b; q^{-1})_{\rho_1 + r}}{(q)_\rho} \prod_{i=1}^{j-1} < u_{r+\rho_i}/u_r > \\ &\quad \times \sum_{\sigma} \frac{1}{(q)_\sigma} \left(< 1/u_r > + \frac{1}{(-q)_{\sigma_{l-j}}} \right) \prod_{i=1}^{l-j} < u_{\sigma_i}/u_r > . \end{aligned}$$

In view of (51) the inner sum over σ is equal to $b_r(z,r)/(q^2,q^2)_r$, and applying (53) we get

$$\begin{aligned} c_r \;&=\; (-1)^r q^{r+2\binom{r}{2}} \frac{(-z/q)_{2r}}{(z^2 q^{2r-2}, q^2)_r} \frac{(a, b; q^{-1})_r}{(q^2; q^2)_r} \\ &\quad \times \sum_{\rho} \frac{(aq^{-r}, bq^{-r}; q^{-1})_{\rho_1}}{(q)_\rho} \prod_{i=1}^{j-1} < u_{r+\rho_i}/u_r > . \end{aligned}$$

Now, the sum over ρ can be computed using (56) with a, b and z replaced by aq^{-r}, bq^{-r} and zq^{2r} respectively. After simplification, we obtain

$$c_r = (-1)^r q^{r+2\binom{r}{2}} \frac{(-z/q)_\infty}{(z^2q^{2r-2}, q^2)_\infty} \frac{(a, b; q^{-1})_r}{(q^2; q^2)_r} \frac{(azq^r,\, bzq^r)_\infty}{(abz)_\infty},$$

which completes the proof.

6 Concluding remarks

Another powerful technique for discovering and proving q-series identities is "Bailey chains", a term coined by Andrews [2]. As in [5], il would be possible to derive our multiple series identities of Rogers-Ramanujan type by this approach, the details are left for interested readers.

Acknowledgement

The second and third authors are partially supported by EC's IHRP Programme, within the Research Training Network "Algebraic Combinatorics in Europe", grant HPRN-CT-2001-00272.

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