A Pfaffian–Hafnian Analogue of Borchardt's Identity

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Abstract

We prove

$$\mathrm{Pf}\left(\frac{x_i - x_j}{(x_i + x_j)^2}\right)_{1\le i,j\le 2n} = \prod_{1\le i<j\le 2n} \frac{x_i - x_j}{x_i + x_j} \cdot \mathrm{Hf}\left(\frac{1}{x_i + x_j}\right)_{1\le i,j\le 2n}$$

(and its variants) by using the complex analysis. This identity can be regarded as a Pfaffian–Hafnian analogue of Borchardt's identity and as a generalization of Schur's identity.

1 Introduction

Determinant and Pfaffian identities play a key role in combinatorics and the representation theory (see, for example, [4], [5], [6], [8], [10], [11]). Among such determinant identities, the central ones are Cauchy's determinant identities ([2])

$$\det\left(\frac{1}{x_i + y_j}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{i,j=1}^{n}(x_i + y_j)}, \tag{1}$$

$$\det\left(\frac{1}{1 - x_i y_j}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{i,j=1}^{n}(1 - x_i y_j)}. \tag{2}$$

C. W. Borchardt [1] gave a generalization of Cauchy's identities:

$$\det\left(\frac{1}{(x_i + y_j)^2}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{i,j=1}^{n}(x_i + y_j)} \cdot \mathrm{perm}\left(\frac{1}{x_i + y_j}\right)_{1\le i,j\le n}, \tag{3}$$

$$\det\left(\frac{1}{(1 - x_i y_j)^2}\right)_{1\le i,j\le n} = \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{i,j=1}^{n}(1 - x_i y_j)} \cdot \mathrm{perm}\left(\frac{1}{1 - x_i y_j}\right)_{1\le i,j\le n}. \tag{4}$$

Here $\mathrm{perm}\, A$ is the permanent of a square matrix A defined by

$$\mathrm{perm}\, A = \sum_{\sigma\in\mathcal{S}_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

This identity (3) is used when we evaluate the determinants appearing in the 0-enumeration of alternating sign matrices (see [11]).

I. Schur [12] gave a Pfaffian analogue of Cauchy's identity (1) in his study of projective representations of the symmetric groups. Schur's Pfaffian identity and its variant ([9], [14]) are

$$\mathrm{Pf}\left(\frac{x_j - x_i}{x_j + x_i}\right)_{1\le i,j\le 2n} = \prod_{1\le i<j\le 2n} \frac{x_j - x_i}{x_j + x_i}, \tag{5}$$

$$\mathrm{Pf}\left(\frac{x_j - x_i}{1 - x_i x_j}\right)_{1\le i,j\le 2n} = \prod_{1\le i<j\le 2n} \frac{x_j - x_i}{1 - x_i x_j}. \tag{6}$$

In this note, we give identities which can be regarded as Pfaffian analogues of Borchardt's identities (3), (4) and as generalizations of Schur's identities (5), (6).

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Theorem 1.1. Let n be a positive integer. Then we have

$$\operatorname{Pf}\left(\frac{x_i - x_j}{(x_i + x_j)^2}\right)_{1\le i,j\le 2n} = \prod_{1\le i<j\le 2n} \frac{x_i - x_j}{x_i + x_j} \cdot \operatorname{Hf}\left(\frac{1}{x_i + x_j}\right)_{1\le i,j\le 2n}, \tag{7}$$

$$\operatorname{Pf}\left(\frac{x_i - x_j}{(1 - x_i x_j)^2}\right)_{1\le i,j\le 2n} = \prod_{1\le i<j\le 2n} \frac{x_i - x_j}{1 - x_i x_j} \cdot \operatorname{Hf}\left(\frac{1}{1 - x_i x_j}\right)_{1\le i,j\le 2n}. \tag{8}$$

Here $\operatorname{Hf} A$ denotes the Hafnian of a symmetric matrix A defined by

$$\operatorname{Hf} A = \sum_{\sigma\in\mathcal{F}_{2n}} a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2n-1)\sigma(2n)},$$

where $\mathcal{F}_{2n}$ is the set of all permutations σ satisfying $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$ and $\sigma(2i-1) < \sigma(2i)$ for $1 \le i \le n$.

2 Proof

In this section, we prove the identity (7) in Theorem 1.1 by using the complex analysis. The other identity (8) is shown by the same method, and also derived from more general identity (18) in Theorem 3.2, which follows from (7). So we omit the proof of (8) here.

Hereafter we put

$$A = \left(\frac{x_i - x_j}{(x_i + x_j)^2}\right)_{1\le i,j\le 2n}, \quad B = \left(\frac{1}{x_i + x_j}\right)_{1\le i,j\le 2n}.$$

For an $2n \times 2n$ symmetric (or skew-symmetric) matrix $M = (m_{ij})$ and distinct indices $i_1, \cdots, i_r$, we denote by $M^{i_1,\cdots,i_r}$ the $(2n-r)\times(2n-r)$ matrix obtained by removing the rows and columns indexed by $i_1, \cdots, i_r$.

First we show two lemmas by using the complex analysis.

Lemma 2.1.

$$\sum_{\substack{1\le k,l\le 2n\\ k\ne l}} \frac{1}{(x_k - z)(x_l + z)} \operatorname{Hf}(B^{k,l}) = \operatorname{Hf}(B) \cdot \sum_{k=1}^{2n} \frac{2x_k}{x_k^2 - z^2}. \tag{9}$$

Proof. Let us denote by $F(z)$ (resp. $G(z)$) the left (resp. right) hand side of (9), and regard $F(z)$ and $G(z)$ as rational functions in the complex variable z, where $x_1, \cdots, x_{2n}$ are distinct complex numbers. Then $F(z)$ and $G(z)$ have poles at $z = \pm x_1, \cdots, \pm x_{2n}$ of order 1. The residues of $F(z)$ at $z = \pm x_m$ are given by

$$\operatorname{Res}_{z=x_m} F(z) = -\sum_{\substack{1\le l\le 2n\\ l\ne m}} \frac{1}{x_l + x_m} \operatorname{Hf}(B^{m,l}), \quad \operatorname{Res}_{z=-x_m} F(z) = \sum_{\substack{1\le k\le 2n\\ k\ne m}} \frac{1}{x_k + x_m} \operatorname{Hf}(B^{k,m}).$$

By considering the expansion of $\operatorname{Hf}(B)$ along the mth row/column, we have

$$\operatorname{Res}_{z=x_m} F(z) = -\operatorname{Hf}(B), \quad \operatorname{Res}_{z=-x_m} F(z) = \operatorname{Hf}(B).$$

On the other hand, the residues of $G(z)$ at $z = \pm x_m$ are given by

$$\operatorname{Res}_{z=x_m} G(z) = -\operatorname{Hf}(B) \cdot \frac{2x_m}{2x_m} = -\operatorname{Hf}(B),$$
$$\operatorname{Res}_{z=-x_m} G(z) = \operatorname{Hf}(B) \cdot \frac{2x_m}{2x_m} = \operatorname{Hf}(B).$$

Since $\lim_{z\to\infty} F(z) = \lim_{z\to\infty} G(z) = 0$, we conclude that $F(z) = G(z)$. □

Lemma 2.2. If n is a positive integer, then

$$\sum_{k=1}^{2n-1} \frac{x_k - z}{(x_k+z)^2} \prod_{\substack{1\le i\le 2n-1\\ i\neq k}} \frac{x_k + x_i}{x_k - x_i} \cdot \mathrm{Hf}(B^{k,2n}) = \prod_{i=1}^{2n-1} \frac{x_i - z}{x_i + z} \sum_{k=1}^{2n-1} \frac{1}{x_k+z} \mathrm{Hf}(B^{k,2n}). \tag{10}$$

Proof. Let $P(z)$ (resp. $Q(z)$) be the left (resp. right) hand side of (10), and regard $P(z)$ and $Q(z)$ as rational functions in z, where $x_1, \cdots, x_{2n-1}$ are distinct complex numbers. Then $P(z)$ and $Q(z)$ have poles at $z = -x_1, \cdots, -x_{2n-1}$ of order 2. Thus, for a fixed m such that $1 \le m \le 2n-1$, we can write

$$P(z) = \frac{p_2}{(z+x_m)^2} + \frac{p_1}{z+x_m} + O(z+x_m),$$
$$Q(z) = \frac{q_2}{(z+x_m)^2} + \frac{q_1}{z+x_m} + O(z+x_m),$$

in a neighborhood of $z = -x_m$. Now we compute the coefficients p_2, p_1, q_2 and q_1, and prove $p_2 = q_2$, $p_1 = q_1$.

By using the relation

$$\frac{x_m - z}{(x_m+z)^2} = \frac{2x_m}{(x_m+z)^2} - \frac{1}{x_m+z},$$

we see that

$$p_2 = 2x_m \prod_{\substack{1\le i\le 2n-1\\ i\neq m}} \frac{x_m + x_i}{x_m - x_i} \cdot \mathrm{Hf}(B^{m,2n}), \tag{11}$$

$$p_1 = -\prod_{\substack{1\le i\le 2n-1\\ i\neq m}} \frac{x_m + x_i}{x_m - x_i} \cdot \mathrm{Hf}(B^{m,2n}). \tag{12}$$

Next we deal with

$$Q(z) = \frac{x_m - z}{x_m + z} \times \prod_{\substack{1\le i\le 2n-1\\ i\neq m}} \frac{x_i - z}{x_i + z} \times \sum_{k=1}^{2n-1} \frac{1}{x_k+z} \mathrm{Hf}(B^{k,2n}).$$

The first factor can be written in the form

$$\frac{x_m - z}{x_m + z} = \frac{2x_m}{x_m+z} - 1.$$

By using the Taylor expansion $\log(1-t) = -t + O(t^2)$, we have

$$\log \frac{x_i - z}{x_m + x_i} = -\frac{z + x_m}{x_i + x_m} + O\left((z+x_m)^2\right),$$
$$\log \frac{x_i + z}{x_m - x_i} = \frac{z + x_m}{x_i - x_m} + O\left((z+x_m)^2\right).$$

Hence we see that

$$\log\left(\frac{x_i - z}{x_i + z} \Big/ \frac{x_i + x_m}{x_i - x_m}\right) = -\frac{2x_i}{x_i^2 - x_m^2}(z+x_m) + O\left((z+x_m)^2\right).$$

Therefore the second factor of $Q(z)$ has the form

$$\prod_{\substack{1\le i\le 2n-1\\ i\neq m}} \frac{x_i - z}{x_i + z} = \prod_{\substack{1\le i\le 2n-1\\ i\neq m}} \frac{x_i + x_m}{x_i - x_m} \cdot \left\{ 1 - \sum_{\substack{1\le k\le 2n-1\\ k\neq m}} \frac{2x_k}{x_k^2 - x_m^2} \cdot (z+x_m) + O\left((z+x_m)^2\right) \right\}.$$

Since we have

$$\frac{1}{x_k+z} = \frac{1}{x_k - x_m} + O\left(z + x_m\right),$$

the last factor of $Q(z)$ has the following expansion:

$$\sum_{k=1}^{2n-1} \frac{1}{x_k + z} \operatorname{Hf}(B^{k,2n}) = \frac{1}{x_m + z} \operatorname{Hf}(B^{m,2n}) + \sum_{\substack{1 \le k \le 2n-1 \\ k \ne m}} \frac{1}{x_k - x_m} \operatorname{Hf}(B^{k,2n}) + O(z + x_m).$$

Combining these expansions, we have

$$q_2 = 2x_m \prod_{\substack{1 \le i \le 2n-1 \\ i \ne m}} \frac{x_i + x_m}{x_i - x_m} \cdot \operatorname{Hf}(B^{m,2n}), \tag{13}$$

and

$$\begin{aligned} q_1 = &\prod_{\substack{1 \le i \le 2n-1 \\ i \ne m}} \frac{x_i + x_m}{x_i - x_m} \\ &\times \left\{ 2x_m \sum_{\substack{1 \le k \le 2n-1 \\ k \ne m}} \frac{\operatorname{Hf}(B^{k,2n})}{x_k - x_m} - 2x_m \operatorname{Hf}(B^{m,2n}) \sum_{\substack{1 \le k \le 2n-1 \\ k \ne m}} \frac{2x_k}{x_k^2 - x_m^2} - \operatorname{Hf}(B^{m,2n}) \right\}. \end{aligned} \tag{14}$$

It follows from (11) and (13) that $p_2 = q_2$. From (12) and (14), in order to prove the equality $p_1 = q_1$, it is enough to show that

$$\sum_{\substack{1 \le k \le 2n-1 \\ k \ne m}} \frac{1}{x_k - x_m} \operatorname{Hf}(B^{k,2n}) = \operatorname{Hf}(B^{m,2n}) \sum_{\substack{1 \le k \le 2n-1 \\ k \ne m}} \frac{2x_k}{x_k^2 - x_m^2}.$$

By permuting the variables $x_1, \cdots, x_{2n-1}$, we may assume that $m = 2n-1$. Then, by expanding the Hafnian on the left hand side along the last row/column, it is enough to show that

$$\sum_{k=1}^{2n-2} \frac{1}{x_k - x_{2n-1}} \sum_{\substack{1 \le l \le 2n-2 \\ l \ne k}} \frac{1}{x_l + x_{2n-1}} \operatorname{Hf}(B^{k,l,2n-1,n}) = \operatorname{Hf}(B^{2n-1,2n}) \sum_{k=1}^{2n-2} \frac{2x_k}{x_k^2 - x_{2n-1}^2}.$$

This follows from Lemma 2.1 (with $2n$ replaced by $2n-2$ and z replaced by x_{2n-1}), and we complete the proof of Lemma 2.2. □

Now we are in the position to prove the identity (7) in Theorem 1.1.

Proof of (7). We proceed by induction on n.

Expanding the Pfaffian along the last row/column and using the induction hypothesis, we see

$$\begin{aligned} \operatorname{Pf}(A) &= \sum_{k=1}^{2n-1} (-1)^{k-1} \frac{x_k - x_{2n}}{(x_k + x_{2n})^2} \operatorname{Pf}(A^{k,2n}) \\ &= \sum_{k=1}^{2n-1} (-1)^{k-1} \frac{x_k - x_{2n}}{(x_k + x_{2n})^2} \prod_{\substack{1 \le i < j \le 2n-1 \\ i,j \ne k}} \frac{x_i - x_j}{x_i + x_j} \operatorname{Hf}(B^{k,2n}). \end{aligned}$$

By using the relation

$$\prod_{\substack{1 \le i < j \le 2n-1 \\ i,j \ne k}} \frac{x_i - x_j}{x_i + x_j} = (-1)^{k-1} \prod_{1 \le i < j \le 2n-1} \frac{x_i - x_j}{x_i + x_j} \cdot \prod_{\substack{1 \le i \le 2n-1 \\ i \ne k}} \frac{x_k + x_i}{x_k - x_i},$$

we have

$$\operatorname{Pf}(A) = \prod_{1 \le i < j \le 2n-1} \frac{x_i - x_j}{x_i + x_j} \sum_{k=1}^{2n-1} \frac{x_k - x_{2n}}{(x_k + x_{2n})^2} \prod_{\substack{1 \le i \le 2n-1 \\ i \ne k}} \frac{x_k + x_i}{x_k - x_i} \cdot \operatorname{Hf}(B^{k,2n}).$$

On the other hand, by expanding the Hafnian along the last row/column, we have

$$\prod_{1\le i<j\le 2n} \frac{x_i - x_j}{x_i + x_j} \cdot \mathrm{Hf}(B) = \prod_{1\le i<j\le 2n} \frac{x_i - x_j}{x_i + x_j} \sum_{k=1}^{2n-1} \frac{1}{x_k + x_{2n}} \cdot \mathrm{Hf}(B^{k,2n}).$$

So it is enough to show the following identity:

$$\sum_{k=1}^{2n-1} \frac{x_k - x_{2n}}{(x_k + x_{2n})^2} \prod_{\substack{1\le i\le 2n-1\\ i\ne k}} \frac{x_k + x_i}{x_k - x_i} \cdot \mathrm{Hf}(B^{k,2n}) = \prod_{i=1}^{2n-1} \frac{x_i - x_{2n}}{x_i + x_{2n}} \sum_{k=1}^{2n-1} \frac{1}{x_k + x_{2n}} \cdot \mathrm{Hf}(B^{k,2n}).$$

This identity follows from Lemma 2.2 and the proof completes. □

3 Generalization

The Cauchy's identities (1) and (2), and the Borchardt's identities (3) and (4) are respectively unified in the following form.

Theorem 3.1. Let $f(x, y) = axy + bx + cy + d$ be a nonzero polynomial. Then we have

$$\det\left(\frac{1}{f(x_i, y_j)}\right)_{1\le i,j\le n} = (-1)^{n(n-1)}(ad - bc)^{n(n-1)/2} \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{1\le i,j\le n} f(x_i, y_j)}, \tag{15}$$

$$\det\left(\frac{1}{f(x_i, y_j)^2}\right)_{1\le i,j\le n} = (-1)^{n(n-1)}(ad - bc)^{n(n-1)/2} \frac{\prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i)}{\prod_{1\le i,j\le n} f(x_i, y_j)} \cdot \mathrm{perm}\left(\frac{1}{f(x_i, y_j)}\right)_{1\le i,j\le n}. \tag{16}$$

Similarly we can generalize the Schur's identities (5) and (6), and our identities (7) and (8).

Theorem 3.2. Let $g(x, y) = axy + b(x + y) + c$ be a nonzero polynomial. Then we have

$$\mathrm{Pf}\left(\frac{x_j - x_i}{g(x_i, x_j)}\right)_{1\le i,j\le 2n} = (b^2 - ac)^{n(n-1)} \prod_{1\le i<j\le 2n} \frac{x_j - x_i}{g(x_i, x_j)}, \tag{17}$$

$$\mathrm{Pf}\left(\frac{x_j - x_i}{g(x_i, x_j)^2}\right)_{1\le i,j\le 2n} = (b^2 - ac)^{n(n-1)} \prod_{1\le i<j\le 2n} \frac{x_j - x_i}{g(x_i, x_j)} \mathrm{Hf}\left(\frac{1}{g(x_i, x_j)}\right)_{1\le i,j\le 2n}. \tag{18}$$

This generalization (17) is given in [7].

Proof. We derive (17) and (18) from (5) and (7) respectively.

First we consider the case where $b^2 - ac \ne 0$. Suppose that $a \ne 0$. Then, by putting

$$A = \frac{1}{2}, \quad B = \frac{1}{2a}(b + \sqrt{b^2 - ac}), \quad C = a, \quad D = b - \sqrt{b^2 - ac},$$

and substituting

$$x_i \to \frac{Ax_i + B}{Cx_i + D} \quad (1 \le i \le 2n)$$

in (5) and (7), we obtain (17) and (18). Similarly we can show the case where $c \ne 0$.

If $b^2 - ac = 0$ and $a \ne 0$, then we have

$$g(x_i, x_j) = a^{-1}(ax_i + b)(ax_j + b).$$

Hence we can evaluate the left hand sides of (17) and (18) by using

$$\mathrm{Pf}\,(x_j - x_i)_{1\le i,j\le 2n} = \begin{cases} x_2 - x_1 & \text{if } n = 1, \\ 0 & \text{if } n \ge 2, \end{cases}$$

and obtain the equalities in (17) and (18). Similarly we can show the case where $b^2 - ac = 0$ and $c \ne 0$. □

From (15) and (16), we have

$$\det\left(\frac{1}{f(x_i,y_j)^2}\right)_{1\le i,j\le n} = \det\left(\frac{1}{f(x_i,y_j)}\right)_{1\le i,j\le n} \cdot \operatorname{perm}\left(\frac{1}{f(x_i,y_j)}\right)_{1\le i,j\le n}.$$

Since the matrix $(f(x_i,y_j))_{1\le i,j\le n}$ has rank at most 2, this identity is the special case of the following theorem.

Theorem 3.3. (Carlitz and Levine [3]) Let $A = (a_{ij})$ be a matrix of rank at most 2. If $a_{ij} \neq 0$ for all i and j, we have

$$\det\left(\frac{1}{a_{ij}^2}\right)_{1\le i,j\le n} = \det\left(\frac{1}{a_{ij}}\right)_{1\le i,j\le n} \cdot \operatorname{perm}\left(\frac{1}{a_{ij}}\right)_{1\le i,j\le n}.$$

From (17) and (18), we have

$$\operatorname{Pf}\left(\frac{x_j - x_i}{g(x_i,x_j)^2}\right)_{1\le i,j\le 2n} = \operatorname{Pf}\left(\frac{x_j - x_i}{g(x_i,x_j)}\right)_{1\le i,j\le 2n} \cdot \operatorname{Hf}\left(\frac{1}{g(x_i,x_j)}\right)_{1\le i,j\le 2n}.$$

It is a natural problem to find a Pfaffian–Hafnian analogue of Theorem 3.3. Also it is interesting to find more examples of a skew-symmetric matrix X and a symmetric matrix Y satisfying

$$\operatorname{Pf}(x_{ij}y_{ij})_{1\le i,j\le 2n} = \operatorname{Pf}(x_{ij})_{1\le i,j\le 2n} \cdot \operatorname{Hf}(y_{ij})_{1\le i,j\le 2n}.$$

Recently there appeared a bijective proof of Borchardt's identity (see [13]). It will be an interesting problem to give a bijective proof of (7) and (8).

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