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# A characterization of the simply-laced FC-finite Coxeter groups

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**Abstract.** *We call an element of a Coxeter group fully covering if its length is equal to the number of the elements covered by it. For the Coxeter groups of type $A$, an element is fully covering if and only if it is 321-avoiding. In this sense it can be regarded as an extended notion of 321-avoiding. It also can be seen from the definition that a fully covering element is always fully commutative. Also, we call a Coxeter group bi-full when an element of the group is fully commutative if and only if it is fully covering. We show that the bi-full Coxeter groups are of type $A$, $D$, $E$. Note that we do not restrict the type $E$ to $E_6, E_7$, and $E_8$. In other words, Coxeter groups of type $E_9, E_{10}, \dots$ are also bi-full. According to a result of Fan, a Coxeter group is a simply-laced FC-finite Coxeter group if and only if it is a bi-full Coxeter group.*

## 1. Introduction

It is needless to say that the notion of Coxeter groups appears in various mathematical fields and have widely interested people, but they, themselves, are still very interesting objects for study. It is also well known that the Coxeter groups of type $A$, $D$, $E_6, E_7$, and $E_8$, i.e. simply laced Weyl groups, share a lot of interesting properties and attract many researchers. Usually, when we say the groups of type $E_n$, we often tend to restrict ourselves to $n = 6, 7$ and 8 cases. However, we sometimes find that the general Coxeter groups of type $E_n$, which are not restricted to $n = 6, 7, 8$, also share some very interesting properties. For example, we can mention FC-finite Coxeter groups. An element of a Coxeter group is said to be *fully commutative* if any reduced expression for it can be obtained from any other by transposing adjacent commuting generators. A *FC-finite* Coxeter group is, by definition, a Coxeter group which has a *finite* number of fully commutative elements. C. K. Fan proved that the simply-laced FC-finite irreducible Coxeter groups are only of type $A, D$, and $E$, and vice versa ([**3**, Proposition 2.]). Here, of course, the Coxeter groups of type $E$ means of type $E_n$, which are not restricted to $n = 6, 7, 8$.

In this paper, we call an element of a Coxeter group *fully covering* if its length equals the number of elements covered by it. This notion was already appeared in [**4**, Theorem 1]. Further we say a Coxeter group is *bi-full* when each element of the group is fully commutative if and only if it is fully covering. The purpose of our paper is to characterize the bi-full Coxeter groups. Although it is a consequence of the result by Fan, that the Coxeter groups of type $A, D, E_6, E_7$, and $E_8$ are bi-full (see [**4**, Theorem 1]) and the Coxeter group of type $\tilde{A}_2$ is not bi-full (see [**4**, Conclusion]), his results do not give a complete characterization of all the bi-full Coxeter groups. In fact, one of our main goals in this paper is to prove that the irreducible bi-full

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Coxeter groups are only of type $A, D, E$ and vice versa. Accordingly it immediately implies that a Coxeter group is a simply-laced FC-finite if and only if it is bi-full (see Theorem 2.9).

Now we recall some notation from the symmetric groups. An element $\sigma$ of the symmetric group of degree $n$ is called a *321-avoiding* if there is no triple $1 \leq i < j < k \leq n$ such that $\sigma(i) > \sigma(j) > \sigma(k)$. Our original motivation is to regard the notion of "fully covering" as an extension of that of 321-avoiding (see [**1**]) to any Coxeter groups. In fact, it is a consequence of some well known facts that, if we restrict our attention to the Coxeter groups of type $A$, then a permutation is fully covering if and only if it is 321-avoiding. Actually, this observation was the starting point of our research. We should note that there is another interesting extension of the notion of 321-avoiding. In [**5**], Green extended the notion to the affine permutation groups from another point of view, whereas our extension, i.e. fully covering, and his definition of 321-avoiding in the affine permutation groups are not equivalent. Indeed, he defined the notion of 321-avoiding permutations for any affine permutation groups and showed that an element is 321-avoiding if and only if it is fully commutative. In [**6**, Thm. 5.1] Hagiwara proved that a 321-avoiding permutation in an affine permutation group is a minuscule element of the group, and vice versa. Meanwhile, it is not hard to see that, for the affine permutation groups, fully covering implies fully commutative, but the reverse is not true.

We conclude this section by making a remark on the Kazhdan-Lusztig theory. Let $W$ be any Coxeter group and let $x, w$ be elements of $W$. Let $p_1(x, w)$ denote the coefficient of degree 1 in the Kazhdan-Lusztig polynomial $P_{x,w}$ for the interval $[x, w]$ in the Bruhat ordering of $W$. M. Dyer showed that $p_1(e, w) = c^-(w) - |\mathrm{supp}(w)|$ and $p_1(e, w) \geq 0$ (see [**2**]). Thus, if $W$ is of type $A$, $D$, $E$ and $w$ is a fully commutative element of $W$, then we can rewrite this result as $p_1(e, w) = \ell(w) - |\mathrm{supp}(w)|$.

This paper is organized as follows: In §2, we recall and provide some basic terminology. In §3, we collect some important properties of a fully commutative element. In §4, we show that Coxeter groups of type $A$, $D$, and $E$ are bi-full. Moreover, we show that a Coxeter group which is neither of type $A, D$ nor $E$ cannot be bi-full.

## 2. Preliminaries and Notations

Now we start with notation and preliminaries again from scratch as this paper become more comprehensive even though it might be slightly repetitious. Throughout this paper, we assume that $(W, S)$ always denote a *Coxeter system* with finite *generator set* $S$ and *Coxeter matrix* $M = [m(s,t)]_{s,t \in S}$. Thus $m(s, t)$ is the order of $st$ in $W$ (possibly $m(s, t) = \infty$). When $m(s, t) = 2$, we say $s$ and $t$ commute. *The Coxeter graph* $\Gamma$ of $(W, S)$ is, by definition, the simple graph with vertex set $S$ and edges between two non-commuting generators. We may regard $(\Gamma, M)$ as a weighted graph by interpreting the entries of $M$ as a weight function on the edges of $\Gamma$, and call it *the Coxeter diagram* of $(W, S)$. We illustrate a Coxeter diagram by labeling an edge $(s, t)$ of the Coxeter graph $\Gamma$ with $m(s, t)$ when $m(s, t) \geq 4$.

We denote the set of integers by $\mathbb{Z}$ and denote the set of positive integers by $\mathbb{Z}_{>0}$. For a positive integer $n$, we put $[n] := \{1, 2, \ldots, n\}$. For a set $A$, we denote its cardinality by $|A|$ or $\sharp A$.

**Notation 2.1.** Let $w$ be an element of $W$ and let $e$ be the identity of $W$. A *length function* $\ell$ is a mapping from $W$ to $\mathbb{Z}$ defined by $\ell(e)$ equals 0 and $\ell(w)$ equals the smallest $m$ such that there exist elements $s_1, s_2, \ldots, s_m$ of $S$ satisfying $w = s_1 s_2 \ldots s_m$ for $w \neq e$. We call $l(w)$ the *length* of $w$. Let $x_1, x_2, \ldots, x_m$ be elements of $W$. If we have $w = x_1 x_2 \ldots x_m$ and $\ell(x_1 x_2 \ldots x_m) = \ell(x_1) + \ell(x_2) + \cdots + \ell(x_m)$, then we call $(x_1, x_2, \ldots, x_m)$ an *extended reduced word* for $w$ and $w = x_1 x_2 \ldots x_m$ an *extended reduced expression* for $w$. Note that we do not assume that $x_1, x_2, \ldots, x_m$ belong to $S$. In particular, we call the word $(x_1, x_2, \ldots, x_m)$ *a reduced word for* $w$ and $w = x_1 x_2 \ldots x_m$ *a reduced expression for* $w$ if $x_i \in S$ for $i = 1, 2, \ldots, m$ and $\ell(w) = m$.

**Definition 2.2.** Let $(W, S)$ and $M = [m(s,t)]_{s,t \in S}$ be as above.

(i) If $\{m(s,t)|s,t \in S\} \subseteq \{1,2,3\}$, then we say $(W,S)$ (resp. $W$) is a *simply-laced* Coxeter system (resp. a simply-laced Coxeter group).
(ii) If there exist elements $s_1, s_2, \ldots, s_m$ of $S$ ($m \geq 3$) such that $m(s_m, s_1) \geq 3$, $m(s_i, s_{i+1}) \geq 3$ for all $i \in [m-1]$, then we say $(W,S)$ (and $W$) is *cyclic*. If not, then we say it is *acyclic*.
(iii) If the Coxeter graph $\Gamma$ of $(W,S)$ is connected, then we say $(W,S)$ (and $W$) is *irreducible*.

**Definition 2.3.** Let $(W,S)$ be a Coxeter system whose Coxeter diagram is given by Figure 1 (resp. Figure 2). Then we call $(W,S)$ a Coxeter system of type $E_{r+4}$ for $r \geq 2$ (resp. type $D_{r+3}$ for $r \geq 1$).

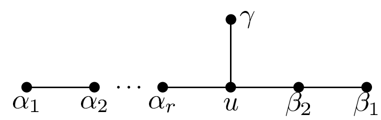


FIGURE 1. Coxeter diagram of type $E_{r+4}$

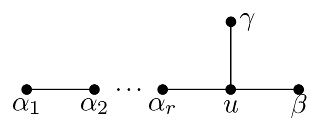


FIGURE 2. Coxeter diagram of type $D_{r+3}$

For integers $m \geq 0$ and $s,t \in S$, set $\langle s,t \rangle_m$ to be the word $\underbrace{(s,t,s,t,s,\ldots)}_{m}$ of length $m$. We introduce an equivalence relation $\approx$ between the words of $S$ generated by the braid relations $\langle s,t \rangle_{m(s,t)} \approx \langle t,s \rangle_{m(s,t)}$ for all $s,t \in S$ such that $m(s,t) < \infty$. It is an important fact that any reduced word for $w$ can be obtained from any other by the braid relations, i.e. the set of reduced words for $w$ consists of one equivalence class with respect to $\approx$. Following [**9**], we also introduce a weaker equivalence relation $\sim$ on the set of the words of $S$ generated by the relations $(s,t) \sim (t,s)$ for all $s,t \in S$ such that $m(s,t) = 2$. We say that $w$ is *fully commutative* if the set of reduced words for $w$ consists of just one equivalence class with respect to $\sim$, i.e. any reduced word for $w$ can be obtained from any other by transposing adjacent commuting pairs. For a Coxeter group $W$, we put

$$W^{FC} := \{w \in W | w \text{ is fully commutative}\}.$$

If the cardinality of $W^{FC}$ is finite, then we say $(W,S)$ is (resp. $W$) a *FC-finite* Coxeter system (resp. a FC-finite Coxeter group).

From now on, we denote a Coxeter group of type $X$ by $W(X)$.

**Theorem 2.4** (C. K. Fan). *If $W$ is an irreducible simply-laced FC-finite Coxeter group, then $W$ should be one of $W(A_n)$, $W(D_{n+3})$ and $W(E_{n+5})$ for some $n \geq 1$ (see [**3**]).*

We recall the definition of the Bruhat ordering. Let $T := \{wsw^{-1} | s \in S, w \in W\}$ be the set of reflections in $W$. Write $y \to z$ if $z = yt$ for some $t \in T$ with $\ell(y) < \ell(z)$. Then define $y < z$ if there is a sequence $y = w_0 \to w_1 \to \cdots \to w_m = z$. It is clear that the resulting relation $y \leq z$ is a partial ordering of $W$, and we call it the *Bruhat ordering*. We say $z$ *covers* $y$ (or equivalently $y$ is covered by $z$), denote by $y \lessdot z$, if $y < z$ and $\ell(y) = \ell(z) - 1$.

The following is a well known characterization of the Bruhat ordering which is called the *subword property*. Give a reduced expression $w = s_1 s_2 \cdots s_m$ for $w \in W$, let us call the products (not necessarily reduced, and possibly empty) of the form $s_{i_1} s_{i_2} \ldots s_{i_q}$ ($1 \leq i_1 < i_2 < \cdots < i_q \leq m$) *the subexpressions of*

$s_1 s_2 \cdots s_m$. Let $w = s_1 \ldots s_m$ be a fixed, but arbitrary reduced expression for $w \in W$. Then $x \leq w$ if and only if $x$ can be obtained as a subexpression of this reduced expression.

The ordering handled in this paper is always assumed to be the Bruhat ordering.

**Notation 2.5.** For $w \in W$, we put

$$\begin{aligned} \mathrm{supp}(w): &= \{s \in S | s \leq w\}, \\ C^-(w): &= \{x \in W | x \lessdot w\}, \\ c^-(w): &= |C^-(w)|. \end{aligned}$$

**Definition 2.6.** For $w \in W$, we call $w$ *fully covering* if $\ell(w) = c^-(w)$.

By the above subword property, the reader easily see that $w \in W$ is fully covering if and only if, given any reduced expression $w = s_1 \cdots s_m$, deleting any one generator from this expression always reduce its length by 1.

If $w \in W$ is not fully commutative, then there exists a reduced expression $w = s_1 \ldots s_m$ including a braid relation $\langle s, t\rangle_{m(s,t)}$ with $m(s,t) \geq 3$. Thus, by discarding one of $s$ or $t$ from this braid relation, we obtain an element $w' < w$ of the form $w' = s_1 \ldots \widehat{s_i} \ldots s_m$ which is not reduced. This immediately shows $w$ is not fully covering, and implies the following proposition.

**Proposition 2.7.** *A fully covering element $w$ of $W$ is fully commutative.*

The reverse is not always true, and we will give some examples later. If the reverse is true, i.e. a fully commutative element $w \in W$ is always fully covering, we say $W$ (resp. $(W, S)$) is a *bi-full* Coxeter group (resp. a *bi-full* Coxeter system).

**Remark 2.8.** Let $(W_1, S_1), (W_2, S_2)$ be bi-full Coxeter systems (resp. FC-finite Coxeter systems). If we have $S_1 \cap S_2 = \emptyset$ and $s_1 s_2 = s_2 s_1$ for any $(s_1, s_2) \in S_1 \times S_2$ then $(W_1 W_2, S_1 \cup S_2)$ is also a bi-full Coxeter system (resp. a FC-finite Coxeter system).

The main result of this paper is the following theorem.

**Theorem 2.9.** *$W$ is a simply-laced FC-finite Coxeter group if and only if $W$ is a bi-full Coxeter group.*

By Remark 2.8, we can easily reduce Theorem 2.9 to the irreducible cases. By Theorem 2.4, we already know that an irreducible simply-laced FC-finite Coxeter group must be one of type $A$, $D$ or $E$. Thus it is enough to show the following theorem to complete the proof of Theorem 2.9.

**Theorem 2.10.** *Let $W$ be an irreducible Coxeter group. Then, $W$ is bi-full if and only if it is either of type $A$, $D$ or $E$.*

By Proposition 2.7, if the following two claims hold then we can obtain Theorem 2.10.

**Claim 1.** Any fully commutative element of the Coxeter group of type $E$ is fully covering (Theorem 4.3).

**Claim 2.** If $W$ is neither of type $A, D$ nor $E$, then there is an element in $W$ which is fully commutative, but not fully covering (Theorem 4.9).

We often use the following notation and facts which the reader may be already familiar with (see [**8**]). For any subset $J \subset S$, let $W_J = \langle J\rangle$ denote the subgroup of $W$ generated by all $s \in J$, which is usually called *the parabolic subgroup of $W$ generated by $J$*. Put $W^J = \{x \in W | \ell(xy) = \ell(x) + \ell(y) \text{ for all } y \in W_J\}$ and ${}^J W = \{x \in W | \ell(yx) = \ell(y) + \ell(x) \text{ for all } y \in W_J\}$, then the following fact shows $W^J$ (resp. ${}^J W$) is the set of left (resp. right) coset representatives of $W$ with respect to $W_J$.

**Fact 2.11.** (i) For any $w \in W$, there is a unique pair $(x, y) \in W^J \times W_J$ such that $w = xy$.
(ii) For any $w \in W$, there is a unique pair $(y, z) \in W_J \times {}^J W$ such that $w = yz$.

## 3. Properties of fully commutative elements

In this section, we collect some basic and important properties of fully commutative elements which will be concerned with the rest of the paper. Throughout this section we assume that $W$ always denotes any Coxeter group if there is no special mention.

By the definition of the fully commutativity, we have the following.

**Lemma 3.1.**

(i) *Let $w$ be an element of $W$. Let $s_1 s_2 \dots s_m$ and $s'_1 s'_2 \dots s'_m$ be reduced expressions for $w$. If $w$ is fully commutative then we have*

$$\{s_1, s_2, \dots, s_m\} = \{s'_1, s'_2, \dots, s'_m\} \text{ as multisets.}$$

(ii) *Assume $m(s,t)$ is odd or 2 for any $s, t \in S$. For any $w \in W$, $w$ is fully commutative if and only if we have $\{s_1, s_2, \dots, s_m\} = \{s'_1, s'_2, \dots, s'_m\}$ as multisets for any reduced expressions $w = s_1 s_2 \dots s_m = s'_1 s'_2 \dots s'_m$.*

(iii) *An element is fully commutative if it has a unique reduced expression.*

(iv) *Let $xyz$ be an extended reduced expression for $w$. If $w$ is fully commutative then $y$ is also fully commutative.*

(v) *Let $W$ be a simply-laced Coxeter group and let $w$ be an element of $W$. Then $w$ is not fully commutative if and only if there is a reduced expression $s_1 s_2 \dots s_m$ for $w$ such that $s_i = s_{i+2}$ for some $1 \le i \le m-2$.*

The following lemma is a key lemma of this paper.

**Lemma 3.2.** *Let $w$ be a fully commutative element and let $s_1 s_2 \dots s_r$ be a reduced expression for $w$ ($r \ge 2$). If we have $w = s s_1 s_2 \dots s_{r-1}$ for some $s \in S$ then we have the followings:*

(i) $s = s_r$,

(ii) $s s_j = s_j s$ *for any* $j \in [r-1]$,

(iii) $s \not\le s_1 s_2 \dots s_{r-1}$.

The following corollary is useful to find an element which is fully commutative and is not fully covering.

**Corollary 3.3.** *Let $w$ be an element of $W$ and let $s_1, s_2, \dots, s_m$ be elements of $S$ such that $w = s_1 s_2 \dots s_m$. Note that we do not assume that $s_1 s_2 \dots s_m$ is a reduced expression for $w$. We define a condition (FC) as follows:*

(FC) *If there exists a pair $(i, j)$ of integers such that $i < j$ and $s_i = s_j$, then there exists a pair $(a, b)$ of integers such that $i < a < b < j$, $s_a s_i \ne s_i s_a$ and $s_b s_i \ne s_i s_b$.*

*Then we have the followings.*

(i) *If $s_1 s_2 \dots s_m$ satisfies the condition (FC) then $s_1 s_2 \dots s_m$ is a reduced expression for $w$ and $w$ is fully commutative.*

(ii) *If $W$ is a simply-laced Coxeter group, $s_1 s_2 \dots s_m$ is a reduced expression for $w$ and $w$ is fully commutative, then $s_1 s_2 \dots s_m$ satisfies the condition (FC).*

By Corollary 3.3, we have the following.

**Corollary 3.4.** *Let $W$ be a simply-laced Coxeter group and let $w$ be an element of $W$ such that $\ell(w^2) = 2\ell(w)$ and $w^2$ is fully commutative. Then for any $k \in \mathbb{Z}_{>0}$ we have $\ell(w^k) = k\ell(w)$ and $w^k$ is fully commutative. In particular, $W$ is not a FC-finite Coxeter group.*

The following lemma holds for any Coxeter system $(W, S)$.

**Lemma 3.5.** *Let $(W, S)$ be a Coxeter system and let $x$ be an element of $W$. Let $s_1$, $s_2$ be elements of $S$ such that $s_1 s_2 x$ is an extended reduced expression and that $s_2 s_1 s_2$ is reduced (i.e. $m(s_1, s_2) \ge 3$). If we have $s_1 \notin supp(x)$ then $s_2 s_1 s_2 x$ is an extended reduced expression.*

The following lemma holds for any simply-laced Coxeter system.

**Lemma 3.6.** *Let $(W, S)$ be a simply-laced Coxeter system, and let $w$ be a fully commutative element of $W$. If $s_1 s_2 \dots s_m$ is a reduced expression for $w$, then $s_1 \widehat{s_2} s_3 \dots s_m$ is reduced.*

## 4. Main results

There is a method to derive the following proposition from a well-known fact on 321-avoiding permutations of the symmetric groups. However, here we give a sketch of our proof without the notion of 321-avoiding.

**Proposition 4.1.** *Let $W$ be a Weyl group of type $A_n$. Then a fully commutative element $w$ of $W$ is fully covering.*

Let $(s_1, s_2, \ldots, s_m) \in S^*$ be any word from $S$ (i.e. an element of the free monoid generated by $S$), and let $\alpha$ be an element of $S$. Then we use the notation:

$$g_\alpha((s_1, s_2, \ldots, s_m)) := \sharp\{i \in [m] \mid s_i = \alpha\}.$$

By Lemma 3.1(i), when $w$ is fully commutative, we can define

$$g_\alpha(w) := g_\alpha((s_1, s_2, \ldots, s_m))$$

without ambiguity where $(s_1, s_2, \ldots, s_m)$ is a reduced word for $w$.

**Lemma 4.2.** *Let $w = s_1 s_2 \ldots s_m$ be a reduced expression for $w \in W$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be a subset of supp($w$) satisfying the following conditions (1),(2), and (3).*

(1) *$\alpha_i s = s\alpha_i$ for any $i \in [r]$ and for any $s \in$ supp$(w) - \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$.*

(2) *$\langle \alpha_1, \alpha_2, \ldots, \alpha_r \rangle$ is a Weyl group of type $A_r$, whose Coxeter graph is given by Figure 3. (i.e. $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ is a connected component of the Coxeter graph of $W$ and $\alpha_1$ is one of its endpoints.)*

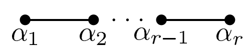


FIGURE 3. Coxeter diagram of type $A_r$

(3) $g_{\alpha_1}((s_1, s_2, \ldots, s_m)) \geq 2$.

*Then $w$ is not fully commutative.*

Here we don't have enough space to give a detailed proof of this lemma, which the reader can find in our original paper [**7**]. The proof of Proposition 4.1 reduce to this lemma. This fact was the starting point of our main result. In fact the idea of the proof of the following theorem resides in a similar method for type $E$, while the proof needs more complicated computations. Thus it is worth describing the proof in the case of type $A$.

**Theorem 4.3.** *Let $W$ be a Coxeter group of type $E$ and let $w$ be an element of $W$. If $w$ is fully commutative then $w$ is fully covering.*

By a similar argument as above, the proof of Theorem 4.3 reduce to the following two lemmas, which is a fundamental idea of our proof. Thus the proofs of the following lemmas are main goal of our paper.

**Lemma 4.4.** *Let $(W, S)$ be a Coxeter system of type $D_{r+3}$, whose Coxeter graph is given by Figure 2 ($r \geq 1$). (i.e. $\alpha_1$, $\beta$ and $\gamma$ are the endpoints designated in the figure.) Put $J := S - \{\alpha_1\}$. Let $w \in {}^J W$ be a fully commutative element and let $s_1 s_2 \ldots s_m$ be a reduced expression for $w$. If supp($w$) includes the endpoints $\alpha_1, \beta, \gamma$, then the followings hold.*

(i) *$r + 3 \leq m$, $s_1 s_2 \ldots s_{r+3} = \alpha_1 \alpha_2 \ldots \alpha_r u \beta \gamma$.*

(ii) *For any $s \in J$, $sw$ is not fully commutative.*

(iii) *$m \leq 2r + 4$.*

(iv) *If $m \geq r + 4$ then we have $s_{r+4} s_{r+5} \ldots s_m = u \alpha_r \alpha_{r-1} \ldots \alpha_{2r+5-m}$ where $\alpha_{r+1} = u$.*

**Lemma 4.5.** *Let $(W, S)$ be a Coxeter system of type $E_{r+4}$ ($r \geq 1$) whose Coxeter graph is designated in Figure 1 (i.e. $\alpha_1$, $\beta_1$ and $\gamma$ are the endpoints in the figure). Put $J := S - \{\alpha_1\}$. Let $w \in {}^J W$ be a fully commutative element and let $s_1 s_2 \ldots s_m$ be a reduced expression for $w$. Then the followings hold.*

(i) *If supp(w) includes all the end points* $\alpha_1$, $\beta_1$ *and* $\gamma$, *then sw is not fully commutative for all* $s \in J$.

(ii) *Assume that we have* $\alpha_1, \beta_2, \gamma \in$ *supp(w),* $\beta_1 \notin$ *supp(w) and* $s \in J$. *If sw is fully commutative then we have* $s = \beta_1$.

(iii) *Assume that we have* $g_{\alpha_1}(w) \geq 2$ *and we have* $s \in J$ *such that sw is fully commutative. Then we have* $w = \alpha_1\alpha_2 \ldots \alpha_r u\gamma\beta_2 u\alpha_r \ldots \alpha_2\alpha_1$ *and* $s = \beta_1$.

(iv) *Assume that we have* $g_{\alpha_1}(w) \geq 3$ *and we have* $w \in {}^J W \cap W^J$. *Then there exists an element* $v$ *of* $W_{S-\{\alpha_1,\alpha_2\}}$ *such that*

$$(\alpha_1\alpha_2 \ldots \alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1} \ldots \alpha_2)\alpha_1\beta_1 v\beta_1(\alpha_2 \ldots \alpha_r u\gamma\beta_2 u\alpha_r\alpha_{r-1} \ldots \alpha_1)$$

*is an extended reduced expression for* $w$ *and that* $\beta_1 v\beta_1 \in {}^{S-\{\beta_1\}}W \cap W^{S-\{\beta_1\}}$.

**Remark 4.6.** Let $w$ be an element of a Coxeter group. In [**4**], $w$ is said to be *short-braid avoiding* if and only if any reduced expression $s_1 s_2 \ldots s_m$ for $w$ satisfies $s_i \neq s_{i+2}$ for all $i \in [m-2]$. It is easy to see that a fully covering element is short-braid avoiding, and that a short-braid avoiding element is fully commutative. By the same method as the one adopted in the proof of [**4**, Theorem 1] and Theorem 4.3, we can easily obtain the following which includes Fan's result [**4**, Theorem 1]. Let $(W, S)$ be a Coxeter system and let $(W_0, S_0)$ be a Coxeter system defined by $S_0 := S$ as a set and $m(s,t) := 3$ if $m(s,t) \geq 3$ in $W$ for $s, t \in S_0$. If $W_0$ is a Coxeter group of type $A$, $D$ or $E$ then for $w \in W$, $w$ is a short-braid avoiding element if and only if $w$ is a fully covering element.

Although it is already shown by Fan that a Coxeter group of type $E$ is FC-finite, we can give an explicit upper bound for the maximum length of fully commutative elements.

**Proposition 4.7.** *For* $n \geq 3$, *we have*

$$\max\{\ell(w) | w \in W(E_n)^{FC}\} \leq 2^{n-1} - 1,$$

*where we put* $W(E_3) := \langle \beta_1, \beta_2, \gamma \rangle$. *In particular, we have* $|W(E_n)^{FC}| < \infty$.

**Remark 4.8.** In [**10**], H. Tagawa showed that we have $\max\{c^-(x) | x \in W(A_n)\} = \lfloor (n+1)^2/4 \rfloor$, where $\lfloor a \rfloor$ is the largest integer equal or less than $a$. By the formula, it is easy to show that we have $\max\{\ell(x) | x \in W(A_n)^{FC}\} = \lfloor (n+1)^2/4 \rfloor$. Note that it does not hold on case of type D. In fact, we have $\max\{c^-(x) | x \in W(D_4)\} = 8 > 6 = \max\{\ell(x) | x \in W(D_4)^{FC}\}$.

Moreover, we can show the following.

**Theorem 4.9.** *Let* $W$ *be an irreducible Coxeter group which is neither of type* $A$, $D$ *nor* $E$. *Then* $W$ *is not a bi-full Coxeter group. In other words, there is an element of* $W$ *which is fully commutative and which is not fully covering. In particular, if* $W$ *is a simply-laced Coxeter group then we have* $|W^{FC}| = \infty$.

This theorem is easily obtained by the following proposition.

**Proposition 4.10.** *Let* $(W_1, S_1)$ *(resp.* $(W_2, S_2)$, $(W_3, S_3)$, $(W_4, S_4)$, $(W_5, S_5)$*) be a Coxeter system of type* $\tilde{A}_n$ $(n \geq 2)$ *(resp.* $\tilde{D}_{r+3}$ $(r \geq 1)$, $\tilde{E}_6$, $\tilde{E}_7$, $I_2(m)$ $(m \geq 4)$*). Then for each* $1 \leq i \leq 5$ *there exists an element* $w_i$ *of* $W_i$ *such that* $w_i$ *is fully commutative and* $w_i$ *is not fully covering. Furthermore we have* $|W_i^{FC}| = \infty$ *for any* $1 \leq i \leq 4$.

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