Zeilberger's Holonomic Ansatz for Pfaffians

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ABSTRACT

A variation of Zeilberger's holonomic ansatz for symbolic determinant evaluations is proposed which is tailored to deal with Pfaffians. The method is also applicable to determinants of skew-symmetric matrices, for which the original approach does not work. As Zeilberger's approach is based on the Laplace expansion (cofactor expansion) of the determinant, we derive our approach from the cofactor expansion of the Pfaffian. To demonstrate the power of our method, we prove, using computer algebra algorithms, some conjectures proposed in the paper "Pfaffian decomposition and a Pfaffian analogue of q-Catalan Hankel determinants" by Ishikawa, Tagawa, and Zeng. A minor summation formula related to partitions and Motzkin paths follows as a corollary.

Categories and Subject Descriptors

G.2.1 [**Discrete Mathematics**]: Combinatorics—*Recurrences and difference equations*; G.4 [**Mathematical Software**]: Algorithm design and analysis

General Terms

Algorithms, Theory

Keywords

Pfaffian, determinant, minor, holonomic systems approach, WZ theory, symbolic summation, computer proof, Motzkin number

1. INTRODUCTION

Pfaffians are a very important concept in combinatorics and in physics, for example, for the enumeration of plane partitions, Kasteleyn's method for the dimer models, etc. We introduce an algorithmic method for evaluating Pfaffians which allows us to solve such problems automatically by computer; we demonstrate its applicability by proving a few conjectures in [6], concerning Pfaffians of interesting combinatorial numbers. Our approach is a variation of Zeilberger's holonomic ansatz for evaluating determinants, which we recall in the following, for sake of self-containedness.

In [17], Zeilberger proposed an algorithmic approach for evaluating and/or producing rigorous proofs of determinant evaluations of the form

$$\det\left(\check{a}_{i,j}\right)_{1\leqslant i,j\leqslant n} = \check{b}_n$$

(we use checked letters here to avoid confusion with the quantities introduced in Section 2). The goal is achieved in a completely automatic fashion, using computer algebra algorithms for guessing recurrences and symbolic summation. The key point is to *guess* [8] a suitable (implicit) description of an auxiliary function $\check{c}_{n,j}$ and then prove that it satisfies the three identities

$$\check{c}_{n,n} = 1 \qquad (n \geqslant 1), \tag{$\check{1}$}$$

$$\sum_{j=1}^{n} \check{c}_{n,j}\check{a}_{i,j} = 0 \qquad (1 \leqslant i < n), \tag{$\check{2}$}$$

$$\sum_{j=1}^{n} \check{c}_{n,j}\check{a}_{n,j} = \frac{\check{b}_n}{\check{b}_{n-1}} \qquad (n \geqslant 1). \tag{$\check{3}$}$$

The determinant evaluation follows as a consequence, using Laplace expansion w.r.t. the last row and induction on n.

In principle, the approach is applicable if the matrix is never singular, i.e., if $\check{b}_n \neq 0$ for all $n \geqslant 0$. But in order to turn Identities ($\check{1}$)–($\check{3}$) into routinely provable tasks, Zeilberger additionally requires that the matrix entries $\check{a}_{i,j}$ constitute a bivariate holonomic sequence and that the ratios of two consecutive determinants $\check{b}_n/\check{b}_{n-1}$ form a univariate holonomic (P-finite) sequence (in other words, $\check{b}_n$ is required to be what one could call *hyper-holonomic*). This is the reason why he termed his approach the *holonomic ansatz* [17, 16]. But still, even if all these conditions are satisfied, Zeilberger's holonomic ansatz is not guaranteed to succeed, because it relies on the fact that the auxiliary function $\check{c}_{n,j}$ (that is, the cofactors of the Laplace expansion with respect to the last row of the $n \times n$ matrix, divided by the determinant of the $(n-1) \times (n-1)$ matrix) turns out to be holonomic, too. This may be the case or not. If one is lucky, i.e., if $\check{c}_{n,j}$ satisfies sufficiently many linear recurrence equations with polynomial coefficients and therefore

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is holonomic, then the holonomic machinery will produce a P-finite recurrence for the sum on the left-hand side of ($\check{3}$). Such a recurrence can then be used to prove a (conjectured) determinant evaluation $\check{b}_n$ by substituting the ratio $\check{b}_n/\check{b}_{n-1}$ into this recurrence and comparing initial values, or even, if the recurrence is not too complicated, to solve it explicitly and obtain a closed form for the determinant.

For a more detailed description and justification of the holonomic ansatz, see [17, 12]. We also recommend the beautiful essay [14] for the reader who is interested in determinant and Pfaffian evaluations in general.

In the present paper, we introduce a variation of Zeilberger's method that is tailored particularly for Pfaffians. Recall that Pfaffians are defined only for skew-symmetric matrices and that the square of the Pfaffian equals the determinant. As a trivial consequence, our approach addresses determinants of skew-symmetric matrices as well. Clearly Zeilberger's holonomic ansatz cannot be applied to skew-symmetric matrices, since the determinant in this case vanishes whenever the dimension is odd. Another extension of Zeilberger's ansatz, the so-called *double-step method*, is applicable to matrices that are zero either for even or odd dimensions [13]. Concerning the evaluation of determinants only (not Pfaffians), the double-step method is more general, as it does not assume skew-symmetry, but at the same time much more complicated and less efficient than our approach for Pfaffians.

In the next section, we state the cofactor expansion of the Pfaffian and use it to develop our algorithmic approach for dealing with evaluations of Pfaffians; this means proof and/or discovery, as in Zeilberger's approach for determinants. In the following Sections 3–5 this method is used to solve some open problems posed in [6]. The details of our computer proofs are provided as supplementary electronic material on the webpage

`http://www.risc.jku.at/people/ckoutsch/pfaffians/`

in form of a Mathematica notebook. It is supposed to enable the reader to reproduce our results and do further experiments. In Section 6 we use our results (Theorem 2) to prove an interesting minor summation formula where the sum ranges over certain partitions and the matrix entries are variations of Motzkin numbers. We conclude this article by posing some open problems as future challenges.

2. PFAFFIANS

Let n be a positive integer and let $A = (a_{i,j})_{1\leqslant i,j\leqslant 2n}$ be a $2n$ by $2n$ skew-symmetric matrix, i.e., $a_{j,i} = -a_{i,j}$, whose entries $a_{i,j}$ are in a commutative ring. Note that it is completely determined by its upper triangular entries $a_{i,j}$ for $1 \leqslant i < j \leqslant 2n$. The *Pfaffian* $\mathrm{Pf}(A)$ of A is defined by

$$\mathrm{Pf}(A) = \sum \epsilon(\sigma_1, \sigma_2, \ldots, \sigma_{2n-1}, \sigma_{2n})\, a_{\sigma_1\sigma_2} \ldots a_{\sigma_{2n-1}\sigma_{2n}}.$$

where the summation is over all partitions

$$\{\{\sigma_1, \sigma_2\}, \ldots, \{\sigma_{2n-1}, \sigma_{2n}\}\}$$

of $[2n] = \{1, 2, \ldots, 2n\}$ into two-elements subsets, and where $\epsilon(\sigma_1, \sigma_2, \ldots, \sigma_{2n-1}, \sigma_{2n})$ denotes the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{2n-1} & \sigma_{2n} \end{pmatrix}.$$

A permutation $(\sigma_1, \sigma_2, \ldots, \sigma_{2n-1}, \sigma_{2n})$ which arises from a partition of $[2n]$ into 2-elements blocks is called a *perfect matching* or a 1-*factor*. For any permutation π of $[2n]$, let $A^\pi = (a_{\pi(i)\pi(j)})$ denote the skew-symmetric matrix obtained by the natural action of π on both rows and columns. From the definition above it is easy to see that

$$\mathrm{Pf}(A^\pi) = \operatorname{sgn}\pi\ \mathrm{Pf}(A).$$

Hence, if any two rows and/or columns are coinciding in A, the Pfaffian $\mathrm{Pf}(A)$ of A vanishes. It is a well-known fact that $\mathrm{Pf}(A)^2 = \det(A)$. Now let $I = \{i_1, \ldots, i_r\}$ be an r-element subset of $[2n]$: we denote by

$$A(I) = A(i_1, \ldots, i_r)$$

the skew-symmetric $(2n-r)\times(2n-r)$ matrix obtained from A by removing the rows $i_1, \ldots, i_r$ and the columns $i_1, \ldots, i_r$. Also let us define $\Gamma_{i,j}$ for $1 \leqslant i, j \leqslant 2n$ by

$$\Gamma_{i,j} = \begin{cases} (-1)^{j-i-1}\,\mathrm{Pf}\,A(i,j) & \text{if } i < j, \\ (-1)^{i-j}\,\mathrm{Pf}\,A(j,i) & \text{if } j < i, \\ 0 & \text{if } i = j. \end{cases}$$

The Laplace expansion formula for Pfaffians reads as follows.

Proposition 1. *Let $A = (a_{i,j})_{1\leqslant i,j\leqslant 2n}$ be a skew-symmetric matrix, and $\Gamma_{i,j}$ be as above. Then we have*

$$\sum_{k=1}^{2n} a_{i,k}\Gamma_{j,k} = \sum_{k=1}^{2n} a_{k,i}\Gamma_{k,j} = \delta_{i,j}\,\mathrm{Pf}\,A.$$

Proof. This statement and its proof are found in [7]. □

Hence if one puts $b_{2n} = \mathrm{Pf}\,A = \mathrm{Pf}(a_{i,j})_{1\leqslant i,j\leqslant 2n}$ and $c_{2n,j} = \Gamma_{j,2n}/\Gamma_{2n-1,2n}$ for $1 \leqslant i \leqslant 2n-1$, then Proposition 1 implies that $c_{2n,j}$ satisfies the following three identities:

$$c_{2n,2n-1} = 1 \qquad (n \geqslant 1), \tag{1}$$

$$\sum_{i=1}^{2n-1} c_{2n,i}a_{i,j} = 0 \qquad (1 \leqslant j < 2n), \tag{2}$$

$$\sum_{i=1}^{2n-1} c_{2n,i}a_{i,2n} = \frac{b_{2n}}{b_{2n-2}} \qquad (n \geqslant 1). \tag{3}$$

Conversely, one easily sees that the bivariate sequence $c_{2n,j}$ is uniquely characterized by Equations (1) and (2), and we can regard (1)–(3) as the formulation analogous to ($\check{1}$)–($\check{3}$) in order to evaluate the Pfaffian $\mathrm{Pf}\,A = \mathrm{Pf}(a_{i,j})_{1\leqslant i,j\leqslant 2n}$. For this purpose, one first has to guess a suitable implicit (i.e., holonomic) description of the function $c_{2n,i}$ and then show that it indeed satisfies the above identities. Induction on n concludes the proof. The methodology is illustrated in detail by an example in Section 3.

Identities (1), (2), and (3) can be proven algorithmically in the spirit of the *holonomic systems approach* [16]. In the following sections the software package `HolonomicFunctions` [11] which runs under the computer algebra system Mathematica is employed for carrying out the necessary computations. The thesis [9] describes the theoretical background and the algorithms implemented therein.

3. A MOTZKIN NUMBER PFAFFIAN

This section gives a detailed computer proof of a Pfaffian involving the Motzkin numbers. It is stated as an open problem in [6], see Formula (6.3) there. The Motzkin numbers M_n can be obtained by the formula

$$M_n = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{2k}\binom{2k}{k} = {}_2F_1\left(\begin{matrix}-\frac{n}{2}, \frac{1-n}{2}\\ 2\end{matrix}; 4\right),$$

where ${}_2F_1$ stands for the Gauß hypergeometric function. They count Motzkin paths from $(0,0)$ to $(n,0)$; recall that a Motzkin path is a path in the lattice $\mathbb{N}_0^2$ that uses only the steps

$$U = (1,1), \qquad H = (1,0), \qquad D = (1,-1)$$

and never runs below the horizontal axis (see [4]).

THEOREM 2. *For all integers $n \geqslant 1$ the following identity holds:*

$$\operatorname{Pf}\left((j-i)M_{i+j-3}\right)_{1\leqslant i,j\leqslant 2n} = \prod_{k=0}^{n-1}(4k+1). \tag{4}$$

PROOF. The proof is split into several parts which are presented in the Sections 3.1–3.5 below. The details of the computations are contained in the supplementary electronic material mentioned in the introduction. ☐

3.1 Implicit Description for $c_{2n,i}$

The first step is to determine the auxiliary function $c_{2n,i}$ that appears in identities (1)–(3), where now

$$a_{i,j} = (j-i)M_{i+j-3}.$$

Using the method of guessing, as implemented in the Mathematica package `Guess` [8], one comes up with an implicit description of this unknown function, namely the following three linear recurrence equations with polynomial coefficients:

$$\begin{aligned}(i-1)&(2n-3)(4n-7)c_{2n,i} = \\ &-(2n+i-4)(8in-8i-8n^2+6n+3)c_{2(n-1),i-1} + \\ &(i-1)(16in-16i+8n^2-34n+27)c_{2(n-1),i} + \\ &24i(i-1)(n-1)c_{2(n-1),i+1} - \\ &(2n-3)(4n-7)(2n-i)c_{2n,i-1},\end{aligned} \tag{5}$$

$$\begin{aligned}(n-2)&(2n-5)(4n-11)(4n-7)(2n-i-2)(2n-i-1)c_{2n,i} = \\ &(2n-5)(4n-11)(8i^2n^2-24i^2n+17i^2-16in^2+48in- \\ &\quad 33i-16n^4+108n^3-258n^2+258n-92)c_{2(n-1),i} - \\ &(n-1)(4n-7)(2n+i-5)(32in^2-122in+ \\ &\quad 117i-32n^3+168n^2-280n+144)c_{2(n-2),i} + \\ &6i(4i+1)(n-2)(n-1)(2n-3)(4n-7)c_{2(n-2),i+1} + \\ &36i(i+1)(n-2)(n-1)(2n-3)(4n-7)c_{2(n-2),i+2},\end{aligned} \tag{6}$$

$$\begin{aligned}18n&(i-3)(i-2)(i-1)c_{2n,i} = \\ &(2n+i-4)(10i^2n-24in^2-63in+i+16n^3+ \\ &\quad 76n^2+97n-3)c_{2n,i-3} - \\ &2(i-3)n(7i^2-12in-46i+33n+73)c_{2n,i-2} - \\ &3(i-3)(i-2)n(14i-12n-39)c_{2n,i-1} - \\ &(2n-1)(4n-3)(2n-i+4)(2n-i+3)c_{2(n+1),i-3}.\end{aligned} \tag{7}$$

When they are rewritten in operator notation, these recurrences form a left Gröbner basis in the corresponding noncommutative operator algebra, which is a bivariate polynomial ring (in the indeterminates S_i and S_n, denoting the forward shift operators w.r.t. i and n, respectively) with coefficients in $\mathbb{Q}(i,n)$. Together with the initial values

$$c_{2,1} = 1, \quad c_{2,2} = c_{2,3} = 0, \quad c_{4,1} = 2,$$

they uniquely define the bivariate sequence $(c_{2n,i})_{n,i\geqslant 1}$. Note that the leading coefficients of (5), (6), and (7) never vanish simultaneously in the region where these recurrences are used to produce the values $c_{2n,i}$ (in the first quadrant, basically).

3.2 Boundary Conditions

The sequence c is now extended to $(c_{2n,i})_{n\geqslant 1, i\in\mathbb{Z}}$ and it is proven that the assumption $c_{2n,i} = 0$ for $i \leqslant 0$ and for $i \geqslant 2n$ is compatible with the recurrences (5)–(7). This knowledge will be useful for the subsequent reasoning.

Provided with the appropriate initial conditions ($c_{2,0} = c_{4,0} = 0$), it is obvious that the recurrence (6) produces zeros on the line $i = 0$, since the terms $c_{2(n-2),i+1}$ and $c_{2(n-2),i+2}$ vanish. Similarly for $i = -1$, since the term $c_{2(n-2),i+2}$ still vanishes; again assuming $c_{2,-1} = c_{4,-1} = 0$. Because of these two zero rows, it is clear that everything beyond them (i.e., for $i < -1$) must be zero as well. A simple computation shows that setting the initial conditions to 0 is compatible with the recurrences (5)–(7).

Since the leading coefficient of (5) does not vanish for any integer point in the area $n \geqslant 2$ and $i \geqslant 2n$, this recurrence can be used to produce the values of $c_{2n,i}$ in this area. The support of (5) indicates that only $c_{2n,2n} = 0$ needs to be shown. The first instances of this sequence are zero by construction, and thus we have just to check that the third-order recurrence (not printed here) that is automatically derived for $c_{2n,2n}$ does not have a singularity in its leading coefficient; this is indeed not the case. The univariate sequence $c_{2n,2n}$ is called the *diagonal* of the bivariate sequence $c_{2n,i}$. Diagonals appear frequently in combinatorial problems and their fast computation is a topic of ongoing research in computer algebra. We used the command `DFiniteSubstitute` of [11] here to perform the substitution $i \to 2n$, which corresponds to the computation of the diagonal.

It remains to show that $c_{2,i} = 0$ for $i > 2$, which is done in a similar fashion.

3.3 Identity (1)

Analogously to the computation of the diagonal in the previous section, an annihilating operator for $c_{2n,2n-1}$ (of order 4, not printed here) is obtained. Its leading coefficient has no nonnegative integer roots, and it has the operator $S_n - 1$ as a right factor. Therefore it annihilates any constant sequence. The four initial values are 1 by construction and therefore $c_{2n,2n-1} = 1$ for all $n \in \mathbb{N}$.

3.4 Identity (2)

Once the implicit descriptions of the bivariate sequences $a_{i,j}$ and $c_{2n,i}$ are available, in terms of zero-dimensional left ideals of recurrence operators, the summation identities (2) and (3) are routinely provable, thanks to software packages like `HolonomicFunctions` [11]. The strategy is as follows: first the closure properties of holonomic functions are employed to compute recurrences for the product $c_{2n,i}a_{i,j}$; the command `DFiniteTimes` does the job. Then the method of creative telescoping is invoked to produce some recurrences for the left-hand side of (2) (this expression is denoted by $g_{n,j}$ in the following). Two different algorithms for this task are implemented in our package, namely the commands `CreativeTelescoping` (Chyzak's algorithm [3]) and `FindCreativeTelescoping` (an alternative ansatz proposed by the second author [10]). In order to prove Identity (2) for instance, some operators of the form

$$P(j,n,S_j,S_n)+(S_i-1)Q(i,j,n,S_i,S_j,S_n)$$

which annihilate the summand $c_{2n,i}a_{i,j}$ are computed. It has already been proven in Section 3.2 that $c_{2n,i}$ is zero outside the summation range which implies that the sum runs over natural boundaries. Therefore the *principal parts* (or *telescopers*, denoted by P above) of the creative telescoping operators annihilate the sum, and the delta parts (denoted by Q) can be disregarded. As a result we find

$$\begin{aligned}&j(4n-7)(2n+j-2)g_{n,j}=\\&\quad j(4n-3)(j-n+1)g_{n-1,j}+\\&\quad (n-1)(4n-3)(2n-j-3)g_{n-1,j+1},\end{aligned}$$

$$\begin{aligned}&(j-2n)(2n+j-2)g_{n,j}=\\&\quad 3(j-2)(j-1)g_{n,j-2}+(j-1)(2j-3)g_{n,j-1}.\end{aligned}$$

A close inspection reveals that only the initial values $g_{1,1}$, $g_{2,1}$, and $g_{2,2}$ need to be given, if the above recurrences shall be used to compute all values of $g_{n,j}$ for $n\geqslant 1$ and $1\leqslant j<2n$. A simple calculation shows that they are all zero, concluding the proof of (2).

Note that the above reasoning is somehow about the maximal possible area: if one tries to extend it further, i.e., to show that $g_{n,j}=0$ in the whole first quadrant, the first step being the points $j=2n$, then the second recurrence, the only one that is applicable in this case, breaks down. Indeed, the values $g_{n,2n}$ are nonzero as is demonstrated in the next section.

3.5 Identity (3)

Identity (3) is done in a very similar fashion, using the method of creative telescoping. Again the summation is over natural boundaries. Thus the principal part of the computed creative telescoping operator gives rise to a recurrence for the left-hand side of (3) which is denoted by r_n here:

$$\begin{aligned}&2(4n-11)(4n-7)(4n-5)(7n-13)r_n=\\&\quad(4n-11)(350n^3-1413n^2+1798n-714)r_{n-1}-\\&\quad 9(n-2)(2n-3)(4n-7)(7n-6)r_{n-2}.\end{aligned}$$

For $n=1$ and $n=2$ the summation in (3) yields the initial values $r_1=1$ and $r_2=5$. It is easily verified that the unique solution of the above recurrence is $r_n=4n-3$. Since $r_n=b_{2n}/b_{2n-2}$ gives the quotients of two consecutive Pfaffians $b_{2n}=\mathrm{Pf}(a_{i,j})_{1\leqslant i,j\leqslant 2n}$, it follows that

$$b_{2n}=\prod_{k=1}^{n}\frac{b_{2k}}{b_{2k-2}}=\prod_{k=1}^{n}(4k-3)=\prod_{k=0}^{n-1}(4k+1).$$

This concludes the proof of Theorem 2.

4. A DELANNOY NUMBER PFAFFIAN

We now consider a Pfaffian that appears as Formula (6.4) in [6], again as an open problem.

THEOREM 3. *Let*

$$D_n=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}$$

denote the n-th central Delannoy number. Then for all integers $n\geqslant 1$ the following identity holds:

$$\mathrm{Pf}\left((j-i)D_{i+j-3}\right)_{1\leqslant i,j\leqslant 2n}=2^{(n+1)(n-1)}(2n-1)\prod_{k=1}^{n-1}(4k-1).\tag{8}$$

PROOF. The auxiliary function $c_{2n,i}$ in this example can be defined by the following recurrences (plus a sufficient amount of initial values):

$$\begin{aligned}&2(i-3)(i-2)(i-1)c_{2n,i}=\\&\quad 3(i-3)(i-2)(8i-27)c_{2n,i-1}-\\&\quad (i-3)(76i^2-589i-8n^2+16n+1109)c_{2n,i-2}+\\&\quad 3(8i^3-105i^2-16in^2+32in+443i+68n^2-\\&\qquad 136n-600)c_{2n,i-3}-\\&\quad (2i-11)(i-2n-3)(i+2n-7)c_{2n,i-4},\end{aligned}$$

$$\begin{aligned}&2(n-2)(2n-3)(4n-9)(i-2n+1)(i-2n+2)c_{2n,i}=\\&\quad (n-1)(i+2n-5)(68i^2n-102i^2-96in^2+178in-\\&\qquad 43i+64n^3-208n^2+200n-56)c_{2(n-1),i}-\\&\quad 6i(n-1)(2n-3)(35i^2+4in-66i-n+14)c_{2(n-1),i+1}+\\&\quad i(i+1)(n-1)(2n-3)(70i+4n-31)c_{2(n-1),i+2}-\\&\quad 6i(i+1)(i+2)(n-1)(2n-3)c_{2(n-1),i+3}.\end{aligned}$$

The proof is very analogous to the one of Theorem 2, see the accompanying Mathematica notebook for the details. □

5. A NARAYANA NUMBER PFAFFIAN

The following Pfaffian appears as Formula (6.6) in [6]:

THEOREM 4. *Let $N_n(x)$ denote the n-th Narayana polynomial defined by*

$$\begin{aligned}N_0(x)&=1,\\N_n(x)&=\sum_{k=0}^{n}\frac{1}{n}\binom{n}{k}\binom{n}{k-1}x^k,\quad(n\geqslant 1).\end{aligned}$$

Then for all integers $n\geqslant 1$ the following identity holds:

$$\mathrm{Pf}\left((j-i)N_{i+j-2}(x)\right)_{1\leqslant i,j\leqslant 2n}=x^{n^2}\prod_{k=0}^{n-1}(4k+1).\tag{9}$$

PROOF. Again, the proof of this evaluation is analogous to the previous ones of (4) and (8), see the accompanying Mathematica notebook for the details. The main difference is that now the free parameter x is involved, which on the one hand makes the computations and the intermediate results more voluminous. On the other hand, some arguments in the proof (like "the leading coefficient of some recurrence is never zero") become more intricate.

One solution to address the latter issue is to argue that x is a formal parameter; then any polynomial in x which is not identically zero, is considered to be nonzero (as an element in the corresponding polynomial ring). If one feels uneasy about this argument, one can as well try to find conditions under which all steps of the proof are sound; for our reasoning the assumption $x<-1$ was sufficient. Hence the evaluation is proven only for $x<-1$. But for specific n, the Pfaffian is a polynomial in x (of a certain degree), as well as the evaluation on the right-hand side of (9). Thus their difference is a polynomial in x which has been proven to be zero for all $x<-1$. By the fundamental theorem of algebra it follows that this polynomial is identically zero, and therefore the evaluation of the Pfaffian is true for all complex numbers x. $\square$

COROLLARY 5. *Let*

$$S_n = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n+k}{2k} \binom{2k}{k}$$

denote the (large) Schröder numbers. Then for all integers $n \geqslant 0$ the following identity holds:

$$\operatorname{Pf}\left((j-i)S_{i+j-2}\right)_{1 \leqslant i,j \leqslant 2n} = 2^{n^2} \prod_{k=0}^{n-1} (4k+1).$$

PROOF. This identity follows from Theorem 4 and the equality $S_n = N_n(2)$; the latter fact can be easily proven from the definitions of these quantities using Zeilberger's algorithm, for example. $\square$

In [6] it has already been noted that Theorem 4 implies the Pfaffian of Corollary 5 involving the Schröder numbers. Similarly, it is stated there that also the Pfaffian (4) is a special case of Theorem 4. However, in order to reflect the historic evolution of our results and for reasons of a clear presentation, we included Theorem 2 and its detailed proof in this article.

6. APPLICATION OF THEOREM 2

Let $A = (a_{i,j})_{1 \leqslant i \leqslant n,\, j \geqslant 1}$ be any n-rowed matrix. If $J = \{j_1, \ldots, j_n\}$ is a set of column indices, then we write $A_J = A_{j_1,\ldots,j_n}$ for the square submatrix of size n obtained from A by choosing the columns indexed by J. If $A = (a_{i,j})_{i,j \geqslant 1}$ is a matrix with infinitely many rows and columns, and $I = \{i_1, \ldots, i_n\}$ (resp. $J = \{j_1, \ldots, j_n\}$) is a set of row (resp. column) indices, then let $A_J^I = A_{j_1,\ldots,j_n}^{i_1,\ldots,i_n}$ denote the square submatrix of size n obtained from A by choosing the rows I and the columns J.

A *partition* is a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers with only finitely many nonzero elements. The number of nonzero elements in λ is called the *length* of λ and is denoted by $l(\lambda)$. An *odd partition* is a partition with odd parts and an *even partition* is a partition with even parts. The *conjugate* of λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$, where λ'_i is the number defined by $\lambda'_i = \#\{j \mid \lambda_j \geqslant i\}$. Given a partition λ such that $l(\lambda) \leqslant n$, let $I_n(\lambda)$ denote the n-element set of nonnegative integers defined by

$$I_n(\lambda) = \{\lambda_n + 1, \lambda_{n-1} + 2, \ldots, \lambda_1 + n\}.$$

For example, $\lambda = (3,3,1,1)$ is an odd partition of length 4, and $I_4(\lambda) = \{2,3,6,7\}$. The conjugate of λ equals $(4,2,2)$, which is an even partition.

Let $H(n) = (h(i,j))_{1 \leqslant i \leqslant n,\, j \geqslant 1}$ denote the n-rowed matrix whose entries are given by

$$h(i, 2k-1) = \binom{i-1}{k-1} {}_2F_1\left(\begin{matrix} \frac{k-i}{2}, \frac{k-i+1}{2} \\ k+1 \end{matrix}; 4\right), \tag{10}$$

$$h(i, 2k) = (i-1) \binom{i-2}{k-1} {}_2F_1\left(\begin{matrix} \frac{k-i+1}{2}, \frac{k-i+2}{2} \\ k+1 \end{matrix}; 4\right). \tag{11}$$

In fact $h(i, 2k-1)$ is the number of Motzkin paths from $(0,0)$ to $(i-1, k-1)$. We also note that $h(i,2k) = k[x^{i+k-1}](1+x+x^2)^{i-1}$, where $[x^n]f(x)$ denotes the coefficient of x^n in a polynomial $f(x)$. For example, if $n = 4$, then we have

$$H(4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 2 & 2 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 4 & 6 & 5 & 6 & 3 & 3 & 1 & 0 & \ldots \end{pmatrix}.$$

For example, $h(4,3) = 5$ gives the number of Motzkin paths from $(0,0)$ to $(3,1)$:

$$UUD, \quad UHH, \quad UDU, \quad HUH, \quad HHU.$$

Meanwhile, $h(4,4) = 6$ equals 2 times the coefficient of x^5 in $(1+x+x^2)^3$. The main purpose of this section is to give a proof of the following theorem as a corollary of Theorem 2.

THEOREM 6. *Let n be a positive integer, and let $H(n)$ be as above. Then we have*

$$\sum_{\substack{\lambda \\ \lambda, \lambda' \text{ even}}} \det H(2n)_{I_{2n}(\lambda)} = \prod_{k=0}^{n-1} (4k+1), \tag{12}$$

where the sum on the left-hand side runs over all even partitions λ such that $\ell(\lambda) \leqslant 2n$ and λ' is also even.

We notice that this theorem is a consequence of an addition formula for ${}_2F_1$ and Theorem 2. But it is not so easy to find a lattice path interpretation of $\det H(2n)_{I_{2n}(\lambda)}$ since we do not know a lattice path interpretation of $h(i, 2k)$. To prove the theorem, we cite the following two lemmas from [5] and [7].

LEMMA 7. *If i and j are nonnegative integers, then we have*

$$\sum_{k \geqslant 0} \binom{i}{k}\binom{j}{k} {}_2F_1\left(\begin{matrix} \frac{k-i+1}{2}, \frac{k-i}{2} \\ k+2 \end{matrix}; 4\right) {}_2F_1\left(\begin{matrix} \frac{k-j+1}{2}, \frac{k-j}{2} \\ k+2 \end{matrix}; 4\right) = {}_2F_1\left(\begin{matrix} \frac{1-i-j}{2}, \frac{-i-j}{2} \\ 2 \end{matrix}; 4\right). \tag{13}$$

PROOF. The proof can be found in [5, Lemma 5.2]. $\square$

LEMMA 8. *For $n \in \mathbb{N}$ let $T = (t_{i,j})_{1 \leqslant i \leqslant 2n,\, j \geqslant 1}$ be an n-rowed matrix, and let $A = (a_{i,j})_{i,j \geqslant 1}$ be a skew-symmetric matrix with infinitely many rows and columns, i.e. $a_{j,i} = -a_{i,j}$ for $i,j \geqslant 1$. Then we have*

$$\sum_{\substack{I \\ \sharp I = 2n}} \operatorname{Pf}(A_I^I) \det(T_I) = \operatorname{Pf}(Q), \tag{14}$$

where the sum on the left-hand side runs over all $2n$-element sets of positive integers and the skew-symmetric matrix Q is defined by $Q = (Q_{i,j}) = TAT^{\mathrm{T}}$ whose entries may be written in the form

$$Q_{i,j} = \sum_{1 \leqslant k < l} a_{k,l} \det(T_{k,l}^{i,j}), \qquad (1 \leqslant i,j \leqslant n). \tag{15}$$

PROOF. The proof of this minor summation formula can be found in [7, Theorem 3.2]. $\square$

PROOF OF THEOREM 6. Set

$$a_{i,j} = \begin{cases} 1 & \text{if } i = 2k-1 \text{ and } j = 2k \text{ for some } k \in \mathbb{N}, \\ -1 & \text{if } i = 2k \text{ and } j = 2k-1 \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

and $t_{i,j} = h(i,j)$ in (14) where $h(i,j)$ is as defined in (10) and (11). Then one can show by direct calculation $Q_{i,j}$ in (15) is given by

$$\begin{aligned}
Q_{i,j} &= \sum_{k\geqslant 1} \det\begin{pmatrix} h(i,2k-1) & h(i,2k) \\ h(j,2k-1) & h(j,2k)\end{pmatrix} \\
&= (j-1)\sum_{k\geqslant 1}\binom{i-1}{k-1}\binom{j-2}{k-1}\,{}_2F_1\left(\begin{matrix}\frac{k-i}{2},\frac{k-i+1}{2}\\ k+1\end{matrix};4\right) \\
&\qquad\qquad \times {}_2F_1\left(\begin{matrix}\frac{k-j+1}{2},\frac{k-j+2}{2}\\ k+1\end{matrix};4\right) - \\
&\quad (i-1)\sum_{k\geqslant 1}\binom{i-2}{k-1}\binom{j-1}{k-1}\,{}_2F_1\left(\begin{matrix}\frac{k-i+1}{2},\frac{k-i+2}{2}\\ k+1\end{matrix};4\right) \\
&\qquad\qquad \times {}_2F_1\left(\begin{matrix}\frac{k-j}{2},\frac{k-j+1}{2}\\ k+1\end{matrix};4\right)
\end{aligned}$$

By (13) we obtain

$$Q_{i,j} = (j-i)\,{}_2F_1\left(\begin{matrix}\frac{3-i-j}{2},\frac{4-i-j}{2}\\ 2\end{matrix};4\right) = (j-i)M_{i+j-3}$$

which, using (14), gives

$$\sum_{\substack{I\\ \#I=2n}} \operatorname{Pf}\big(A_I^I\big)\det H(2n)_I = \operatorname{Pf}\big((j-i)M_{i+j-3}\big)_{1\leqslant i,j\leqslant 2n}.$$

It is not hard to see that $\operatorname{Pf}(A_I^I) = 1$ if $I = I_{2n}(\lambda)$ for a partition λ such that $\ell(\lambda) \leqslant 2n$ and λ, λ' are even, and $\operatorname{Pf}(A_I^I) = 0$ otherwise (see [7]). Hence we obtain the desired formula (12) as a consequence of Theorem 2. □

We can regard the numbers

$$h(i,2k-1) = \binom{i-1}{k-1}\,{}_2F_1\left(\begin{matrix}\frac{k-i}{2},\frac{k-i+1}{2}\\ k+1\end{matrix};4\right)$$

as a generalization of the Motzkin numbers M_n since they count the Motzkin paths from $(0,0)$ to $(i-1,k-1)$, and write $\mathcal{M}_i^{(k)} = h(i,2k-1)$ hereafter. In fact $\big(\mathcal{M}_i^{(k)}\big)_{i\geqslant 1}$ gives the $(k-1)$-th column of the Motzkin triangle [4]. Note that $M_n = \mathcal{M}_{n+1}^{(1)}$ so that Theorem 2 reads

$$\operatorname{Pf}\big((j-i)\mathcal{M}_{i+j-2}^{(1)}\big) = \prod_{k=0}^{n-1}(4k+1).$$

It may now be attractive to present a generalization of Theorem 2 as follows.

Conjecture 9. *Let n and k be positive integers.*

(i) Then the Pfaffian

$$\operatorname{Pf}\Big((j-i)\mathcal{M}_{i+j-2}^{(k)}\Big)_{1\leqslant i,j\leqslant 2n}$$

equals

$$\prod_{i=0}^{m-1}\prod_{j=0}^{k-1}(4ki+2j+k)$$

if $m = n/k$ is an integer, and it equals

$$\left(\prod_{j=1}^{\lfloor k/2\rfloor}\frac{1}{2j-k}\right)\left(\prod_{i=0}^{m-1}\prod_{j=1}^{k}(4ki+2j-k)\right)$$

if k is odd and $m = \big(n+\lfloor k/2\rfloor\big)/k$ is an integer. The Pfaffian is zero in all other cases.

(ii) Meanwhile, the Pfaffian

$$\operatorname{Pf}\Big((j-i)\left(\mathcal{M}_{i+j-2}^{(k)} + \mathcal{M}_{i+j-1}^{(k)}\right)\Big)_{1\leqslant i,j\leqslant 2n}$$

equals

$$\prod_{i=0}^{m-1}\prod_{j=0}^{k-1}(4ki+2j+k+1)$$

if $m = n/k$ is an integer, and it equals

$$\left(\prod_{j=1}^{k/2}\frac{1}{2j-k-1}\right)\left(\prod_{i=0}^{m-1}\prod_{j=1}^{k}(4ki+2j-k-1)\right)$$

if k is even and $m = \big(n+k/2\big)/k$ is an integer. The Pfaffian is zero in all other cases.

We want to remark that for $k=1$ part (i) is just Theorem 2 and part (ii) can be proven analogously. Unfortunately these two Pfaffians are periodically zero if $k \geqslant 2$. This prevents us from applying our method to the conjecture, since we consider the quotient of two consecutive Pfaffians. Of course, one could come up with a Pfaffian analogue of the double-step method presented in [13], which would settle Conjecture 9 for the special case $k=2$. This construction may be extended for $k=3$, $k=4$, etc., at the cost of more and more involved computations. However, this approach will not work for symbolic k in general.

Conjecture 9 can be regarded as a Pfaffian analogue of the Hankel determinants of Motzkin numers [1, Proposition 2], and Hankel determinants of sums of two consecutive Motzkin numbers [2, Theorem 3.2]. Many combinatorial arguments are known for the Hankel determinants, but little is known for Hankel Pfaffians (see [6, 15]). It may be interesting to discover a combinatorial reason why we can expect such a nice formula for the Hankel Pfaffians of (sums of) Motzkin numbers.

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