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New Schur Function Series

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This paper presents some new product identities for certain summations of Schur functions. These identities are generalizations of some famous identities known to Littlewood and appearing in Macdonald's book. We refer to these identities as the "Littlewood-type formulas." In addition, analogues for summations of characters of the other classical groups are also given. The Littlewood-type formulas in this paper are separated into two classes, the rational Schur function series and the generalized Schur function series. An application of a rational Schur function series to the infinite product representation of the elliptic theta functions is also given. We prove these Littlewood-type formulas using the Cauchy–Binet formula. The Cauchy–Binet formula is a basic but powerful tool applicable in the present context, which can be derived from our Pfaffian formula, as we explain.

0. INTRODUCTION

In this paper we present some new product identities for certain summations of Schur functions that generalize some of the results in [IOW, IW2] and [YW, LP]. As elegantly proved in [Ma], the following formulas were

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known to Littlewood:

$$\sum_{\lambda=(\alpha-1|\alpha)} (-1)^{|\lambda|/2} s_\lambda(x) = \prod_{i<j} (1 - x_i x_j)$$

$$\sum_{\lambda=(\alpha|\alpha)} (-1)^{(|\lambda|+p(\lambda))/2} s_\lambda(x) = \prod_{i} (1 - x_i) \prod_{i<j} (1 - x_i x_j)$$

$$\sum_{\lambda=(\alpha+1|\alpha)} (-1)^{|\lambda|/2} s_\lambda(x) = \prod_{i} (1 - x_i^2) \prod_{i<j} (1 - x_i x_j).$$

These formulas are extended in [YW, LP] and are studied as relations among the characters of the general linear groups. Indeed, historically, Schur function identities have been used, to great effect, to obtain compact expressions for the evaluation of branching rules and tensor products for finite-dimensional representations of classical groups. In fact, the study of Schur functions is still an active branch of theoretical physics, through the study on both infinite and finite representations of noncompact groups and representations of compact groups. In this paper we prove several theorems establishing generalizations of Littlewood's formulas and obtain some detailed results derived from these generalized theorems. The main results of this paper are Theorem 2.1 and Theorem 5.1, which are the generalized forms of our results discussed in [IOW]. Theorem 2.9 and Theorems 6.1, 6.2, and 6.3 are the B, C, and D versions of these theorems.

This paper is organized as follows. In Section 1 we give basic notation and definitions. In Section 2 we study Schur function identities and their B, C, and D versions, which assume simple forms. In Section 3 we investigate q-specializations of the formulas obtained in Section 2, and we give an enumerative combinatorial proof of the infinite product representation of the elliptic theta functions, e.g.,

$$\begin{aligned}\vartheta_1(v,\tau) &= 2\sum_{k=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)\pi v \\ &= 2q^{1/4} Q_0 \sin \pi v \prod_{n=1}^{\infty} \left(1 - 2q^{2n} \cos 2\pi v + q^{4n}\right),\end{aligned}$$

for $Q_0 = \prod_{n=1}^{\infty}(1-q^{2n})$. In the final part of Section 3, we make some remarks concerning the theta function from the point of view of representation theory. In Section 4, we prove a combinatorial theorem Theorem 4.1, which will be needed to describe more complicated, generalized

Littlewood-type formulas. In the process of deriving these identities, an argument concerning the relation between (Sato's) Maya diagrams and Young diagrams plays a crucial role. We attempt to construct a reasonably systematic method for using Maya diagrams to evaluate minor determinants of various rectangular matrices in question. Because of its nature, this method is also expected to provide an interesting technique. In Section 5 we give the generalized form of the Littlewood-type formulas. Theorem 5.1 is somewhat abstract and complicated, as it represents the most general form. For this reason we give many concrete identities as the illustrating example of the theorem. In Section 6 we state the B, C, and D versions of Theorem 5.1. Our study of the "Littlewood-type formulas" is stimulated by a Pfaffian formula concerning the summation of determinants obtained in [IW1] (see Theorem 1.1 for details). In this paper we use only the Cauchy–Binet formula, which may seem to be a very simple tool. However, we have found that this simple machinery is powerful enough to prove most of the formulas known collectively as the "Littlewood-type formulas." In a subsequent paper we will investigate the formulas that are obtained essentially not by the Cauchy–Binet formula, but by the Pfaffian formula.

1. BASIC NOTATION AND SUMMATION FORMULA

Let us denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{Z}$ the set of integers. Let $[m]$ denote the subset $\{1, 2, \dots, m\}$ of $\mathbb{N}$ for a positive integer m. Let r, m, and n be positive integers such that $r \leq m, n$, and let T be any m by n matrix. For r-element subsets $I = \{i_1 < \cdots < i_r\}$ and $K = \{k_1, \dots, k_r\}$ of row and column indices, let $T_K^I = T_{k_1 \cdots k_r}^{i_1 \cdots i_r}$ denote the submatrix of T obtained by picking up the rows and columns indexed by I and K. In the case in which I contains all row indices, we omit $I = [m]$ from the above expression and write T_K^I simply as T_K.

Assume $n \leq N$, and let B be an arbitrary n by n skew-symmetric matrix; that is, $B = (b_{ij})$ satisfies $b_{ij} = -b_{ji}$. In Theorem 1 of [IW1], we obtain a result concerning a certain summation of minors that we call the minor summation formula of the Pfaffian. In this paper we use this theorem without proof and deduce the Cauchy–Binet theorem from it.

THEOREM 1.1. *Let $n \leq N$ and assume n is even. Let $T = (t_{ik})$ be any n by N matrix, and let $B = (b_{ik})$ be any N by N skew-symmetric matrix with entries b_{ik}. Then*

$$\sum_{\substack{I \subset [N] \\ \#I = n}} \operatorname{pf}\left(B_I^I\right) \det(T_I) = \operatorname{pf}(Q), \tag{1.1}$$

where Q is the n by n skew-symmetric matrix defined by $Q = TB^tT$, i.e.,

$$Q_{ij} = \sum_{1 \le k < l \le n} b_{kl} \det\left(T_{kl}^{ij}\right), \qquad (1 \le i, j \le m). \tag{1.2}$$

As a corollary of the minor-summation formula given in the above theorem, we obtain the following Cauchy–Binet formula, which provides a basic tool throughout this paper. We have already stated this corollary in [IOW], but here we give another simple proof.

COROLLARY 1.2. *Assume $n \le N$, and let $X = (x_{ik})_{1 \le i \le n,\, 1 \le k \le N}$ and $Y = (y_{ik})_{1 \le i \le n,\, 1 \le k \le N}$ be any n by N matrices. Then*

$$\sum_{\substack{K \subset [N] \\ \#K = n}} \det(X_K)\det(Y_K) = \det(X^tY). \tag{1.3}$$

Proof. We can assume that n is even without loss of generality, since if n is odd, we can take two $(n+1)$ by $(N+1)$ matrices X' and Y' in the form

$$X' = \begin{pmatrix} 1 & O \\ O & X \end{pmatrix} \qquad Y' = \begin{pmatrix} 1 & O \\ O & Y \end{pmatrix},$$

and apply the result for the even case. Fix any skew-symmetric matrix A whose Pfaffian is 1. Apply Theorem 1.1 with $B = {}^tYAY$ and $T = X$. Then we obtain the left-hand side of (1.3). Moreover, if we apply Theorem 1.1 with $B = A$ and $T = X^tY$, then we obtain the right-hand side of (1.3). This completes the proof. ∎

Remark. Cauchy–Lagrange's formula is proved by taking matrices simply as

$$X = Y = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}.$$

Now we define some basic concepts and introduce some necessary notation. A weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1, \dots, \lambda_m)$ with $\lambda_1 \ge \cdots \ge \lambda_m \ge 0$ is called a *partition* of $|\lambda| = \lambda_1 + \cdots + \lambda_m$. The *length* $\ell(\lambda)$ of a partition λ is the number of non-zero terms of λ. If an integer i appears exactly m_i times as a part of λ, we write $\lambda = (1^{m_1} 2^{m_2} \cdots)$. For example,

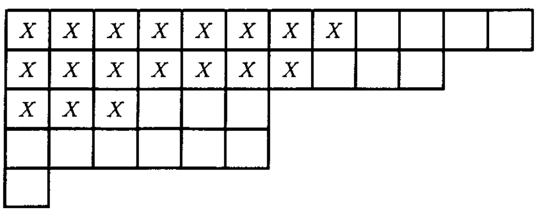
$$(r^n) = (\underbrace{r, r, \dots, r}_{n\text{-times}}).$$

The partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ defined by $\lambda'_i = \#\{j\colon \lambda_j \ge i\}$ is referred to as the *conjugate* partition of λ. Let $n(\lambda) = \sum_{i \ge 1}(i-1)\lambda_i = \sum_{i \ge 1}\binom{\lambda'_i}{2}$. For each cell $x = (i, j)$ in λ, the *hook-length* of λ at x is defined to be $h(x) = \lambda_i - j + \lambda'_j - i + 1$. Suppose that the main diagonal of λ consists of $r = p(\lambda)$ nodes. Then let $\alpha_i = \lambda_i - i$ and $\beta_i = \lambda'_i - i$ for $1 \le i \le r$. We also denote the partition λ by $\lambda = (\alpha_1, \dots, \alpha_r | \beta_1, \dots, \beta_r) = (\alpha | \beta)$. This is called the Frobenius notation. Slightly abusing the Frobenius notation, we may disregard the order of the α_i and β_i and admit α and β to be any r element subsets of $\mathbb{N}$. When α and β are any disjoint subsets of $\mathbb{N}$, let $\#(\alpha < \beta) = \#\{(x, y);\ x \in \alpha,\ y \in \beta,\ x < y\}$. The number $\#(\alpha > \beta)$ can be defined similarly. When α and/or β are each composed of just one element (for example, $\beta = (k)$), we use the notation $\#(\alpha < k)$. For example, $\lambda = (5441)$ is the partition of 14 with $p(\lambda) = 3$. This partition is denoted by $\lambda = (421|310)$ in the Frobenius notation. If $\alpha = \{0, 1, 3\}$, then $\#(\alpha > 2) = 1$, $\#(\alpha < 2) = 2$ and $(\alpha + 1|\alpha) = (421|310)$.

Given a partition λ such that $\ell(\lambda) \le n$, let $J_n(\lambda)$ denote the n-element set $\{\lambda_1 + n - 1, \lambda_2 + n - 2, \dots, \lambda_n\}$. If there is no danger of confusion, we omit n and simply denote this set by $J(\lambda)$.

If $(\alpha|\beta)$ and $(\gamma|\delta)$ are given partitions such that $\alpha \cap \gamma = \emptyset$ and $\beta \cap \delta = \emptyset$, then we denote $(\alpha \cup \gamma|\beta \cup \delta)$ by $(\alpha|\beta) \cup (\gamma|\delta)$. For example, $(421|310) \cup (0|2) = (4210|3210)$.

Let λ and μ be the partitions such that $\lambda \supset \mu$. The set theoretic difference $\theta = \lambda - \mu$ is called the *skew diagram*. Let $\theta_i = \lambda_i - \mu_i$, $\theta'_i = \lambda'_i - \mu'_i$, and $|\theta| = |\lambda| - |\mu|$. We call a skew diagram θ a *horizontal* m-band (resp., a *vertical m-band*) if $\theta_i \le m$ (resp., $\theta'_i \le m$). When $m = 1$, we call this a *horizontal* (resp., *vertical*) *strip*. Fix an integer k such that $0 \le k \le m$. For a horizontal m-band θ', a part θ'_i that is equal to k is called a k-part of θ'. A maximal connected set of k-parts is called a *k-component* of θ'. For example, let $\lambda = (12, 10, 6, 6, 1)$ and $\mu = (8, 7, 3)$ as in the following diagram. Then $\theta' = (2, 1, 1, 2, 2, 2, 0, 1, 2, 2, 1, 1)$, and $(\theta'_4, \theta'_5, \theta'_6)$ is a 2-component of the horizontal 2-band θ'.



For a k-component $c = (\theta'_i, \theta'_{i+1}, \dots, \theta'_j)$ of θ', let $|c|$ denote the number of parts (i.e., $(j - i + 1)$) and call this number the weight of c. Let $C_k(\theta')$ denote the set of k-components of θ'. For example, the above horizontal 2-band contains three 1-components and three 2-components. The 1-components have weights 2, 1, and 2, and the 2-components have weights 1, 3, and 2. For a vertical m-band θ, the k-components and their weights are defined similarly. We also denote the set of k-components by $C_k(\theta)$.

The Schur functions are well-known symmetric functions, which are known as the characters of the irreducible polynomial representations of the general linear group on a torus. For the purpose of describing characters corresponding to the other classical groups, we present additional notation.

We take the following realizations of four series of classical Lie algebras of types A, B, C, and D throughout the paper:

$$\begin{aligned}
\mathfrak{g}_{A(n)} &= \mathfrak{gl}(n, \mathbb{C}), \\
\mathfrak{g}_{B(n)} &= \mathfrak{so}(2n+1, \mathbb{C}) \\
&= \left\{ X \in \mathfrak{gl}(2n+1, \mathbb{C}) \colon J_{\mathfrak{so}(2n+1)} X + {}^t X J_{\mathfrak{so}(2n+1)} = 0 \right\}, \\
\mathfrak{g}_{C(n)} &= \mathfrak{sp}(2n, \mathbb{C}) = \left\{ X \in \mathfrak{gl}(2n, \mathbb{C}) \colon J_{\mathfrak{sp}(2n)} X + {}^t X J_{\mathfrak{sp}(2n)} = 0 \right\}, \\
\mathfrak{g}_{D(n)} &= \mathfrak{so}(2n, \mathbb{C}) = \left\{ X \in \mathfrak{gl}(2n, \mathbb{C}) \colon J_{\mathfrak{so}(2n)} X + {}^t X J_{\mathfrak{so}(2n)} = 0 \right\}.
\end{aligned}$$

Here $J_{\mathfrak{so}(N)}$ and $J_{\mathfrak{sp}(2n)}$ are given by the following antidiagonal matrices:

$$J_{\mathfrak{so}(N)} = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}, \qquad J_{\mathfrak{sp}(2n)} = \begin{pmatrix} 0 & J_{\mathfrak{so}(n)} \\ -J_{\mathfrak{so}(n)} & 0 \end{pmatrix}.$$

Let $\mathfrak{h}_{X(n)}$ be the Cartan subalgebra consisting of diagonal matrices in $\mathfrak{g}_{X(n)}$, where $X(n)$ represents one of $A(n)$, $B(n)$, $C(n)$, and $D(n)$. Also, let $\varepsilon_i \colon \mathfrak{h} \to \mathbb{C}$ be the linear functional assigning the (i, i)-entry of $H \in \mathfrak{h}_{X(n)}$ to H. Then we can take a simple system of roots as

$$\begin{aligned}
\Pi_{A(n)} &= \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n\}, \\
\Pi_{B(n)} &= \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}, \\
\Pi_{C(n)} &= \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}, \\
\Pi_{D(n)} &= \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}.
\end{aligned}$$

The finite-dimensional irreducible representations of $\mathfrak{g}_{X(n)}$ are parameterized by the following dominant integral weights:

$$\begin{aligned}
P^+_{A(n)} &= \{\lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n \colon \lambda_i \in \mathbb{C},\ \lambda_k - \lambda_{i+1} \in \mathbb{N}\}, \\
P^+_{B(n)} &= \{\lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n \colon \lambda \text{ is a partition or a half-partition}\}, \\
P^+_{C(n)} &= \{\lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n \colon \lambda \text{ is a partition}\}, \\
P^+_{D(n)} &= \{\lambda_1 \varepsilon_1 + \cdots + \lambda_{n-1} \varepsilon_{n-1} \pm \lambda_n \varepsilon_n \colon \\
&\qquad \lambda \text{ is a partition or a half-partition}\}.
\end{aligned}$$

Here a half partition λ we mean is a non-increasing sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of nonnegative half integers $\lambda_i \in \mathbb{N} + \frac{1}{2}$. Namely we may write $\lambda = (\mu_1 + \frac{1}{2}, \dots, \mu_n + \frac{1}{2})$ with $\ell(\mu) \le n$. If there is no confusion, we simply write $\lambda = \mu + \frac{1}{2}$.

If $X(n) = A(n)$, $B(n)$, or $C(n)$, and if $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition or a half-partition, we denote by $\lambda_{X(n)}$ the (formal) irreducible character of $\mathfrak{g}_{X(n)}$ with highest weight $\lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n$. In the $D(n)$ case, we define $\lambda^{\pm}_{D(n)}$ to be the irreducible characters of $\mathfrak{so}(2n, \mathbb{C})$ with highest weights $\lambda_1 \varepsilon_1 + \cdots + \lambda_{n-1} \varepsilon_{n-1} \pm \lambda_n \varepsilon_n$, respectively. Note that $\lambda^+_{D(n)} = \lambda^-_{D(n)}$ if $\ell(\lambda) < n$. Here we regard a character as a Laurent polynomial in the variables $x_i^{\pm 1/2} = e^{\pm \varepsilon_i/2}$.

We now recall the Weyl character formula. Let

$$T^{X(n)} = \left(T^{X(n)}_{ik}\right)_{i=1,\dots,n} \qquad (X = A, B, C, D^+, D^-, D)$$

be the n-rowed matrix defined by

$$T^{A(n)}_{ik} = x_i^k, \quad T^{C(n)}_{ik} = x_i^{k+1} - x_i^{-k-1} \qquad (\text{for } k \in \mathbb{N}),$$

$$T^{B(n)}_{ik} = x_i^{k+1/2} - x_i^{-k-1/2}, \quad T^{D^+(n)}_{ik} = x_i^k + x_i^{-k}, \quad T^{D^-(n)}_{ik} = x_i^k - x_i^{-k} \qquad \left(\text{for } k \in \tfrac{1}{2}\mathbb{N}\right).$$

Then the Weyl character formula can be written in the following form; for a partition or a half-partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we have

$$\lambda_{X(n)} = \frac{\det\left(T^{X(n)}_{J(\lambda)}\right)}{\Delta_{X(n)}} \quad (\text{for } X = A, B, C),$$

$$\lambda^{\pm}_{D(n)} = \frac{\det\left(T^{D^+(n)}_{J(\lambda)}\right) \pm \det\left(T^{D^-(n)}_{J(\lambda)}\right)}{\Delta_{D(n)}}.$$

Furthermore, concerning the characters of $D(n)$, we use the notation

$$[\lambda]^{\pm}_{D(n)} = [\lambda_1, \dots, \lambda_N]^{\pm}_{D(n)} = \frac{\det\left(T^{D^{\pm}(n)}_{J(\lambda)}\right)}{\Delta_{D(n)}}$$

in the calculations of the later sections. Here, the so-called Weyl denominator $\Delta_{X(n)} = \det(T^{X(n)}_{J(\emptyset)})$ for $X = A, B, C$, and $\Delta_{D(n)} = \det(T^{D^+(n)}_{J(\emptyset)})$ are explicitly given by

$$\Delta_{A(n)} = \prod_{1 \le i < j \le n} (x_j - x_i) \qquad (\text{the Vandermonde determinant}),$$

$$\Delta_{B(n)} = (-1)^{n(n+1)/2} (x_1 \cdots x_n)^{-n+1/2} \prod_{i=1}^{n} (1 - x_i) \times \prod_{1 \le i < j \le n} (x_j - x_i)(1 - x_i x_j),$$

$$\Delta_{C(n)} = (-1)^{n(n+1)/2}(x_1 \cdots x_n)^{-n} \prod_{i=1}^{n}(1-x_i^2) \times \prod_{1\le i<j\le n}(x_j - x_i)(1-x_ix_j),$$

$$\Delta_{D(n)} = 2(-1)^{n(n-1)/2}(x_1 \cdots x_n)^{-n+1} \prod_{1\le i<j\le n}(x_j - x_i)(1-x_ix_j).$$

Note that only in the case of the $A(n)$ type, we sometimes denote the irreducible character corresponding to a partition (highest weight) λ by s_λ. This is called the Schur function, in place of $\lambda_{A(n)}$. (We sometimes use the abbreviation S-functions for the Schur functions.)

Next we recall the Chebyshev polynomials of the first and second kinds. *The Chebyshev polynomials of the first kind* are defined by $T_n(a) = \cos(n \arccos a)$, and *the Chebyshev polynomials of the second kind* are defined by $U_n(a) = \sin(n \arccos a)/\sin(\arccos a)$. The first few terms of these functions are given by $T_0(a) = 1$, $T_1(a) = a$, $T_2(a) = 2a^2 - 1$, $T_3(a) = 4a^3 - 3a$, $T_4(a) = 8a^4 - 7a^2 + 1$, and $U_1(a) = 1$, $U_2(a) = 2a$, $U_3(a) = 4a^2 - 1$, $U_4(a) = 8a^3 - 4a$. The pairs $(T_n(a), U_n(a))$ satisfy the recurrence formulas

$$\begin{aligned} T_{n+1}(a) &= aT_n(a) + (a^2-1)U_n(a), \\ U_{n+1}(a) &= T_n(a) + aU_n(a), \end{aligned} \tag{1.4}$$

and individually, $T_n(a)$ and $U_n(a)$ satisfy the recurrence formulas

$$\begin{aligned} T_{n+2}(a) - 2aT_{n+1}(a) + T_n(a) &= 0, \\ U_{n+2}(a) - 2aU_{n+1}(a) + U_n(a) &= 0. \end{aligned} \tag{1.5}$$

We can also define $T_n(a)$ and $U_n(a)$ for $n < 0$, as the above recurrence formulas always hold. The generating functions of $T_n(a)$ and $U_n(a)$ are given by

$$\begin{aligned} T_0(a) + 2\sum_{n=0}^{\infty} T_n(a)x^n &= \frac{1-x^2}{1-2ax+x^2}, \\ \sum_{n=0}^{\infty} U_{n+1}(a)x^n &= \frac{1}{1-2ax+x^2}. \end{aligned} \tag{1.6}$$

Put

$$u_{ij}(a) = \begin{cases} 2a & \text{if } i = j, \\ 1 & \text{if } j = i+1 \text{ or } i = j+1, \\ 0 & \text{otherwise}, \end{cases} \tag{1.7}$$

for $i, j \geq 1$. Let $U^{(n)}(a)$ be the n by n matrix whose (i, j)-entry is given by $u_{ij}(a)$. Then it is easy to see that $\det U^{(n-1)}(a) = U_n(a)$, since, if we expand the determinant $\det U^{(n)}(a)$ along the first row, then we see that the polynomials $U_n(a)$ satisfy the above recursion formula.

2. POLYNOMIAL AND RATIONAL SCHUR FUNCTION SERIES

In this section we investigate certain formulas involving the Chebyshev polynomials and the characters of the classical groups. It is also possible to derive these formulas from the Cauchy identity. We also show here that the infinite product representations of the elliptic theta functions are proved as a corollary of our formula by employing a purely combinatorial method.

Let $R(x)$ be a rational function of x. Hereafter we assume that $R(0) = 1$, and put $R(x) = \sum_{i=0}^{\infty} a_i x^i$ with $a_0 = 1$ as a formal power series. Let $M^{R,n}$ denote the n-rowed matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_0 & a_1 & \cdots \end{pmatrix}$$

induced from $R(x)$. We allow n to be ∞. For a given partition λ such that $\ell(\lambda) \leq n$, we define

$$c_\lambda^R = \det M_{J_n(\lambda)}^{R,n}.$$

Note that, by the assumption $a_0 = 1$, $\det M_{J_n(\lambda)}^{R,n}$ is independent of n. Furthermore, we have

$$c_\lambda^R = 0 \qquad \text{if } \ell(\lambda') > d$$

when $R(x)$ is a polynomial of degree d. The following theorem is elementary and is an immediate consequence of the Cauchy–Binet formula.

THEOREM 2.1. *If $R(x) = \sum_{i=0}^{\infty} a_i x^i$ is a rational function such that $a_0 = 1$, then we have*

$$\sum_\lambda c_\lambda^R s_\lambda(x_1, \ldots, x_n) = \prod_{i=1}^{n} R(x_i). \tag{2.1}$$

Proof. Apply the Cauchy–Binet formula to the matrices $M^{R,n}$ and $T^{A(n)}$ defined in Section 1. Then we have

$$\sum_{\lambda} c_{\lambda}^{R} s_{\lambda}(x_1, \dots, x_n) \Delta_{A(n)} = \det M^{R,n\,t}T^{A(n)}.$$

The right-hand side of this identity is equal to

$$\det \operatorname{diag}\bigl(R(x_1), \dots, R(x_n)\bigr) T_{J(\emptyset)}^{A(n)} = \prod_{i=1}^{n} R(x_i) \Delta_{A(n)}.$$

This proves our theorem. ∎

Given a rational function $R(x)$ such that $R(0) = 1$, we can write $\prod_{i=1}^{n} R(x_i)$ as a linear combination of (an infinite number of) the Schur functions by the method described in the above theorem. We refer to this type of S-function series as a *rational* S-function series defined by R. The above theorem reduces the problem of obtaining a rational S-function series to the problem of determining c_{λ}^{R} for a given rational function R. In particular, when R is a polynomial, we call the S-function series a *polynomial* S-function series. In this section we intensively investigate polynomial S-function series. The following lemma gives us a method of obtaining an additional rational S-function series from a known one.

LEMMA 2.2. *Suppose that $R(x)$ is a rational function of x. If we have an identity*

$$\sum_{\lambda} a_{\lambda} s_{\lambda}(x_1, \dots, x_n) = \prod_{i=1}^{n} R(x_i),$$

then we also have

$$\sum_{\lambda} a_{\lambda} s_{\lambda'}(x_1, \dots, x_n) = \prod_{i=1}^{n} R(-x_i)^{-1}. \tag{2.2}$$

Proof. Recall the (graded) ring of symmetric function $\Lambda = \Lambda[x_1, x_2, \dots]$ in countably many independent variables $x_1, x_2, \dots$ and the involution ω of Λ in Chapter I of [Ma]. It is known that images of the Schur function s_{λ} and the mth power sum function p_m under ω are given by

$$\omega(s_{\lambda}) = s_{\lambda'}, \qquad \omega(p_m) = (-1)^{m-1} p_m,$$

where λ' denotes the conjugate partition of λ. Hence it is clear that the left-hand side of the first identity is transformed to that of the second one under ω.

Without loss of generality, we may assume that $R(x)$ has the form $R(x) = P(x)/Q(x)$, where $P(x) = 1 + a_1 x + \cdots + a_m x^m$ and $Q(x) = 1 + b_1 x + \cdots + b_l x^l$. Then using the Maclaurin expansion of $\log(1+x)$, it is easy to see that

$$\log \prod_{i=1}^{n} P(x_i) = \sum_{N \geq 1} \frac{(-1)^{N-1}}{N} \sum_{k_1 + \cdots + k_m = N} \frac{N!}{k_1! \cdots k_m!} a_1^{k_1} \cdots a_m^{k_m} \times p_{k_1 + 2k_2 + \cdots + mk_m}(x_1, \ldots, x_n).$$

It follows immediately from this expression that

$$\omega\left(\log \prod_{i=1}^{n} P(x_i)\right) = -\log \prod_{i=1}^{n} P(-x_i).$$

A similar equation holds for $Q(x)$, and hence one also holds for $R(x)$. This concludes the proof of the lemma. ∎

LEMMA 2.3. *Let $R(x) = \sum_{i=0}^{\infty} a_i x^i$ be a rational function such that $a_0 = 1$. We define the sequence $\{b_n^R\}$ for $n \geq 0$ by $b_n^R = c_{1^n}^R$. Here 1^n represents the partition with n 1's. Then the generating function of b_n^R is given by*

$$\sum_{n=0}^{\infty} b_n^R x^n = \frac{1}{R(-x)}.$$

Proof. Note that b_n^R is defined by the n by n determinant

$$b_n^R = \begin{vmatrix} a_1 & a_2 & & \cdots \\ a_0 & a_1 & & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_0 & a_1 \end{vmatrix}.$$

If we expand this determinant along the first row, we obtain the following identities:

$$\begin{aligned} b_0^R &= 1 \\ b_1^R &= a_1 b_0^R \\ b_2^R &= a_1 b_1^R - a_2 b_0^R \\ &\cdots \\ b_n^P &= a_1 b_{n-1}^R - a_2 b_{n-2}^R + \cdots + (-1)^{n-1} a_n b_0^R \\ &\cdots \end{aligned}$$

By using these identities, we immediately obtain

$$R(-x)\sum_{n=0}^{\infty} b_n^R x^n = 1.$$

This completes the proof. ∎

We define the n-rowed matrix $N^{R,n}$ by

$$N^{R,n} = \begin{pmatrix} b_0^R & b_1^R & b_2^R & b_3^R & & & \cdots \\ 0 & b_0^R & b_1^R & b_2^R & & & \cdots \\ 0 & 0 & b_0^R & b_1^R & & & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & & 0 & b_0^R & b_1^R & \cdots \end{pmatrix}.$$

Since $b_0^R = 1$, if $n \geq \ell(\lambda)$, then $\det N_{J_n(\lambda)}^{R,n}$ is independent of the choice of n. The following lemma is an easily consequence of the above lemmas.

Lemma 2.4. *Let $R(x) = \sum_{i=0}^{\infty} a_i x^i$ be a rational function such that $a_0 = 1$. If λ is a partition and n is a nonnegative integer such that $\ell(\lambda) \leq n$, then*

$$\det N_{J_N(\lambda)}^{(R,n)} = c_{\lambda'}^R.$$

Here λ' represents the conjugate partition of λ.

Lemma 2.5. *Let $S(x)$ be a rational function such that $S(0) = 1$.*

(1) *If $R(x) = S(x)(1 + a_1 x)$, then we have*

$$c_\lambda^R = \sum_\mu a_1^{|\lambda - \mu|} c_\mu^S,$$

where the summation runs over all μ such that $\mu \subset \lambda$ and $\lambda - \mu$ is a vertical strip.

(2) *If $R(x) = S(x)/(1 - a_1 x)$, then we have*

$$c_\lambda^R = \sum_\mu a_1^{|\lambda - \mu|} c_\mu^S,$$

where the summation runs over all μ such that $\mu \subset \lambda$ and $\lambda - \mu$ is a horizontal strip.

(3) *If $R(x) = S(x)(1 + a_1 x + a_2 x^2)$, then we have*

$$c_\lambda^R = \sum_\mu b_{\lambda - \mu} c_\mu^S,$$

where the summation runs over all μ *such that* $\mu \subset \lambda$ *and* $\lambda - \mu$ *is a vertical* 2-*band. Also, for a vertical* 2-*band* θ *let*

$$b_\theta = a_2^{\sum_{\sigma \in C_2(\theta)}|\sigma|} \prod_{\sigma \in C_1(\theta)} d(\sigma)$$

with

$$d(\sigma) = c_{1^{|\sigma|}}^{1+a_1x+a_2x^2} = \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & \cdots & a_2 & a_1 & a_0 \\ 0 & \cdots & 0 & a_2 & a_1 \end{vmatrix}.$$

Here the determinant has size $|\sigma|$, *and* $C_k(\theta)$ *denotes the set of* k-*components of* θ.

(4) *If* $R(x) = S(x)/(1 - a_1x + a_2x^2)$, *then we have*

$$c_\lambda^R = \sum_\mu b'_{\lambda-\mu} c_\mu^S,$$

where the summation runs over all μ *such that* $\mu \subset \lambda$ *and* $\lambda - \mu$ *is a horizontal* 2-*band. Also, for a horizontal* 2-*band* θ', *let*

$$b'_{\theta'} = a_2^{\sum_{\sigma \in C_2(\theta')}|\sigma|} \prod_{\sigma \in C_1(\theta')} d(\sigma).$$

Proof. Assertions (2) and (4) are immediately obtained from (1) and (3). Here we give an outline of the proof of (3). When $R(x) = S(x) \cdot (1 + 2a_1x + a_2x^2)$, note that $M^{R,n} = M^{S,n}M^{1+a_1x+a_2x^2,\infty}$. Let λ be a partition such that $\ell(\lambda) \le n$. Then we see that $M_{J_n(\lambda)}^{R,n}$ is equal to the determinant $\det N^{R,n}\,{}^tW$, where W is the n-rowed matrix of the form

$$W = \begin{pmatrix} a_2 & & & & & \cdots & & & & & 0 \\ a_1 & a_2 & & & & & & & & & \\ 1 & a_1 & a_2 & & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & & \\ & & 1 & a_1 & a_2 & & & & & & \\ & & & 1 & a_1 & a_2 & & & & & \\ \vdots & & & & \ddots & \ddots & \ddots & & & & \vdots \\ & & & & & 1 & a_1 & a_2 & & & \\ & & & & & & 1 & a_1 & a_2 & & \\ & & & & & & & \ddots & \ddots & \ddots & \\ & & & & & & & & 1 & a_1 & a_2 \\ & & & & & & & & & 1 & a_1 \\ 0 & & & & & \cdots & & & & & 1 \end{pmatrix}$$

By inspection of this matrix, we obtain the desired identity by the Cauchy–Binet formula. ∎

As a corollary of the above lemma we obtain the following result. (We can also immediately obtain the conjugate case of this corollary, but we do not give it here.)

COROLLARY 2.6. *Let $S(x)$ be a rational function such that $S(0) = 1$.*

(1) *If $R(x) = S(x)(1 + x)$, then we have*

$$c_\lambda^R = \sum_\mu c_\mu^S,$$

where the summation runs over all μ such that $\mu \subset \lambda$ and $\lambda - \mu$ is a vertical strip.

(2) *If $R(x) = S(x)(1 - x)$, then we have*

$$c_\lambda^R = \sum_\mu (-1)^{|\lambda - \mu|} c_\mu^S,$$

where the summation runs over all μ such that $\mu \subset \lambda$ and $\lambda - \mu$ is a vertical strip.

(3) *If $R(x) = S(x)(1 + 2ax + x^2)$, then we have*

$$c_\lambda^R = \sum_\mu b_{\lambda - \mu} c_\mu^S,$$

where the summation runs over all μ such that $\mu \subset \lambda$ and $\lambda - \mu$ is a vertical 2-band, and

$$b_{\lambda - \mu} = \prod_{\sigma \in C_1(\lambda - \mu)} U_{|\sigma| + 1}(a),$$

where $U_k(a)$ is the kth Chebyshev polynomial of the second kind.

(4) *If $R(x) = S(x)(1 - x^2)$, then we have*

$$c_\lambda^R = \sum_\mu (-1)^{\sum_{\sigma \in C_2(\lambda - \mu)} |\sigma|} c_\mu^S,$$

where the summation runs over all μ such that $\mu \subset \lambda$, and $\lambda - \mu$ is a vertical 2-band such that all the 1-components of $\lambda - \mu$ have even weights.

From this point we restrict our attention to the case in which $R(x)$ is a polynomial. Given a polynomial $P(x) = \sum_{i=0}^d a_i x^i$ of degree d such that $a_0 = 1$, we call $P(x)$ *symmetric* if the coefficients satisfy $a_{d-i} = a_i$ for $0 \le i \le d$ and *antisymmetric* if $a_{d-i} = -a_i$ for $0 \le i \le d$. In this section we have no reason to restrict our attention to symmetric or antisymmetric polynomials, but we will find that symmetric and antisymmetric polynomials play an important role in the following sections. Hence in this paper we focus mainly on symmetric or antisymmetric polynomials.

If we enumerate symmetric and antisymmetric polynomials of small degree d, we obtain Table I. More generally, the following lemma holds.

LEMMA 2.7. *Let $P(x) = \sum_{i=0}^{d} a_i x^i$ be a polynomial of degree d such that $a_0 = 1$. Assume that $P(x)$ is symmetric.*

(1) *If d is even, then $P(x)$ can be written in the form*

$$P(x) = Q(x)(1 + 2ax + x^2),$$

where $Q(x)$ is a symmetric polynomial of degree $d - 2$.

(2) *If d is odd, then $P(x)$ can be written in the form*

$$P(x) = Q(x)(1 + x),$$

where $Q(x)$ is a symmetric polynomial of degree $d - 1$.

Assume that $P(x)$ is antisymmetric.

(1) *If d is even, then $P(x)$ can be written in the form*

$$P(x) = Q(x)(1 - x^2),$$

where $Q(x)$ is a symmetric polynomial of degree $d - 2$.

(2) *If d is odd, then $P(x)$ can be written in the form*

$$P(x) = Q(x)(1 - x),$$

where $Q(x)$ is a symmetric polynomial of degree $d - 1$.

The proof of this lemma is easy and we therefore omit it. This lemma enables us to calculate the c_λ^P recursively. Here we determine all of the c_λ^P

TABLE I

Degree	Symmetric	Antisymmetric
$d = 0$	1	0
$d = 1$	$1 + x$	$1 - x$
$d = 2$	$1 + 2ax + x^2$	$1 - x^2$
$d = 3$	$(1 + x)(1 + 2ax + x^2)$	$(1 - x)(1 + 2ax + x^2)$
$d = 4$	$(1 + 2ax + x^2)(1 + 2bx + x^2)$	$(1 - x^2)(1 + 2ax + x^2)$

corresponding to the polynomials appearing in Table I. When $d = 0$, it is clear that

$$c_\lambda^1 = \begin{cases} 1 & \text{if } \lambda = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

When $d = 1$, it is easy to see that

$$c_\lambda^{1+x} = \begin{cases} 1 & \text{if } \lambda = (1^p), \\ 0 & \text{otherwise,} \end{cases} \qquad c_\lambda^{1-x} = \begin{cases} (-1)^p & \text{if } \lambda = (1^p), \\ 0 & \text{otherwise.} \end{cases}$$

When $d = 2$, we have

$$c_\lambda^{1+2ax+x^2} = \begin{cases} U_{q+1}(a) & \text{if } \lambda = (2^p 1^q), \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

and

$$c_\lambda^{1-x^2} = \begin{cases} (-1)^p & \text{if } \lambda = (2^p 1^q) \text{ with } q \text{ even}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

From these results we obtain the following proposition.

PROPOSITION 2.8. *We have*

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} U_{k+1}(a) s_{(k+l,l)}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{1 - 2ax_i + x_i^2},$$

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} U_{k+1}(a) s_{(2^l 1^k)}(x_1, \ldots, x_n) = \prod_{i=1}^{n} (1 + 2ax_i + x_i^2).$$

Remark. Cauchy's determinant formula,

$$\frac{1}{\Delta_{A(n)}(x)} \frac{1}{\Delta_{A(m)}(y)} \det\left(\frac{1}{1 - x_j y_k} \right) = \prod_{j=1}^{n} \prod_{k=1}^{m} \frac{1}{1 - x_j y_k}$$

is equivalent to the $(GL_m(\mathbb{C}), GL_n(\mathbb{C}))$-dual pair in the sense of Howe. In other words, it implies the irreducible decomposition of the polynomial algebra $\mathscr{P}(\mathrm{Mat}_{m \times n}(\mathbb{C}))$ on the matrix space $\mathrm{Mat}_{m \times n}(\mathbb{C})$ as the $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$-module,

$$\sum_{\substack{\lambda \\ \ell(\lambda) \le \min\{m,n\}}} s_\lambda(y_1, \ldots, y_m) s_\lambda(x_1, \ldots, x_n) = \prod_{j=1}^{n} \prod_{k=1}^{m} \frac{1}{1 - x_j y_k}.$$

Now assume $m = 2$. Then λ in the above sum runs over partitions of the form $\lambda = (l+k, l)$ $(k, l \geq 0)$. Put $y_1 = e^{i\theta}$, $y_2 = y_1^{-1} = e^{-i\theta}$. This procedure simply corresponds to considering the action of the product group $SU_2(\mathbb{C}) \times GL_n(\mathbb{C})$ on $\mathscr{P}(\mathrm{Mat}_{m\times n}(\mathbb{C}))$ in place of $GL_2(\mathbb{C}) \times GL_n(\mathbb{C})$. Since $s_{(l+k,l)}(e^{i\theta}, e^{-i\theta}) = \sin(k+1)\theta / \sin\theta$, we have

$$\sum_{k,l\geq 0} \frac{\sin(k+1)\theta}{\sin\theta} s_{(l+k,l)}(x_1, \dots, x_n) = \prod_{j=1}^{n} \frac{1}{1 - 2\cos\theta x_j + x_j^2}.$$

This obviously gives the first formula in the above proposition.

Now we continue working through the table by considering degree $d = 3$ and $d = 4$. When $d = 3$, the symmetric and antisymmetric cases are as follows.

If $P(x) = (1+x)(1+2ax+x^2)$, then

$$c_\lambda^P = \begin{cases} \displaystyle\sum_{i=0}^{q}\sum_{j=0}^{r} U_{i+j+1}(a) & \text{if } \lambda = (3^p 2^q 1^r), \\ 0 & \text{otherwise}. \end{cases} \tag{2.5}$$

If $P(x) = (1-x)(1+2ax+x^2)$, then

$$c_\lambda^P = \begin{cases} \displaystyle\sum_{i=0}^{q}\sum_{j=0}^{r} (-1)^{p+i+r-j} U_{i+j+1}(a) & \text{if } \lambda = (3^p 2^q 1^r), \\ 0 & \text{otherwise}. \end{cases} \tag{2.6}$$

When $d = 4$, we have the following results.

If $P(x) = (1+2ax+x^2)(1+2bx+x^2)$, then

$$c_\lambda^P = \begin{cases} \displaystyle\sum_{i=0}^{r-1} (r-1-i) U_{i+1}(a) U_{q+1}(b) U_{i+1}(b) U_{s+1}(b) \\ \displaystyle\quad + \sum_{i=0}^{r-1}\sum_{j=0}^{s} U_{i+s-j+1}(a) U_{q+1}(b) U_{i+1}(b) U_{j+1}(b) \\ \displaystyle\quad + \sum_{i=0}^{q}\sum_{j=0}^{r} U_{q-i+j+1}(a) U_{i+1}(b) U_{j+1}(b) U_{s+1}(b) \\ \displaystyle\quad + \sum_{i=0}^{q}\sum_{j=0}^{s-1} U_{q-i+r+s-j+1}(a) U_{i+1}(b) U_{r+1}(b) U_{j+1}(b) \\ \qquad \text{if } \lambda = (4^p 3^q 2^r 1^s), \\ 0 \qquad \text{otherwise}. \end{cases} \tag{2.7}$$

If $P(x) = (1 - x^2)(1 + 2ax + x^2)$, then

$$c_\lambda^P = \begin{cases} (-1)^p \Bigg\{ e(q)e(r)e(s) \displaystyle\sum_{i=0}^{[r/2]-1} U_{2i+1}(a) \\ + e(q) \displaystyle\sum_{\substack{i=0 \\ i \neq r/2}}^{[r/2]} \sum_{j=0}^{[s/2]} U_{2i+s-2j+1}(a) \\ + e(s) \displaystyle\sum_{i=0}^{[q/2]} \sum_{j=0}^{[r/2]} (-1)^{q-2i+r-2j} U_{q-2i+2j+1}(a) \\ + e(r) \displaystyle\sum_{i=0}^{[q/2]} \sum_{\substack{j=0 \\ j \neq s/2}}^{[s/2]} (-1)^{q-2i} U_{q-2i+r+s-2j+1}(a) \Bigg\} & \text{if } \lambda = (4^p 3^q 2^r 1^s), \\ 0 & \text{otherwise}. \end{cases} \tag{2.8}$$

where

$$e(k) = \begin{cases} 1 & \text{if } k \text{ is even}, \\ 0 & \text{if } k \text{ is odd}. \end{cases}$$

The following theorem is the B, C, D version of Theorem 2.1.

THEOREM 2.9. *Let $n \in \mathbb{N}$ and let $X = B$, C, or $D^\pm$. If $P(x) = \sum_{i=0}^d a_i x^i$ is a symmetric polynomial with $a_0 = 1$, then*

$$\sum_{\substack{\lambda \\ \ell(\lambda) \le n}} c_\lambda^P (\lambda + s^n)_{X(n)} = \left(\left(s + \frac{d}{2} \right)^n \right)_{X(n)} \prod_{i=1}^n P(x_i) x_i^{-d/2}. \tag{2.9}$$

Here, $s \in \frac{1}{2}\mathbb{N}$ if $X = B$ or $D^\pm$, while $s \in \mathbb{N}$ and d must be even if $X = C$. If $P(x)$ is an antisymmetric polynomial with $a_0 = 1$, then

$$\sum_{\substack{\lambda \\ \ell(\lambda) \le n}} c_\lambda^P (\lambda + s^n)_{B(n)} = \frac{((s + 1/2 + d/2)^n)_{D^+(n)} \prod_{i=1}^n P(x_i) x_i^{-d/2}}{\prod_{i=1}^n \left(x_i^{1/2} - x_i^{-1/2} \right)},$$

$$\sum_{\substack{\lambda \\ \ell(\lambda) \le n}} c_\lambda^P (\lambda + s^n)_{C(n)} = \frac{((s + 1 + d/2)^n)_{D^+(n)} \prod_{i=1}^n P(x_i) x_i^{-d/2}}{\prod_{i=1}^n \left(x_i - x_i^{-1} \right)},$$

$$\sum_{\substack{\lambda \\ \ell(\lambda) \le n}} c_\lambda^P (\lambda + s^n)_{D^\pm(n)} = \pm \left(\left(s + \frac{d}{2} \right)^n \right)_{D^\pm(n)} \prod_{i=1}^n P(x_i) x_i^{-d/2}.$$

Proof. Let $T^{n,\alpha,\pm} = (T_{ik}^{\alpha,\pm})$ be the n-rowed matrix whose (i,k)-entries are given by

$$T_{ik}^{\alpha,\pm} = x_i^{k+\alpha} \pm x_i^{-k-\alpha} \qquad \text{for } k \in \tfrac{1}{2}\mathbb{N}.$$

We apply the Cauchy–Binet formula to the determinant $M^{P,n\,t}T^{n,s+\alpha,\pm}$. Because of the relation

$$\sum_{k=0}^{d} a_k\left(x_j^{k+i-1+s+\alpha} \pm x_j^{-k-i+1-s-\alpha}\right)$$
$$= P(x_j)\,x_j^{-d/2}\left(x_j^{i+s+\alpha+d/2} \pm x^{-i-s-\alpha-d/2}\right)$$

we have

$$\det\left(M^{P,n\,t}T^{n,s+\alpha,\pm}\right) = \prod_{i=1}^{n} P(x_i)\,x_i^{-d/2} \det T_{J_n(\varnothing)}^{n,s+\alpha+d/2,\pm}.$$

If we put

$$\psi_{\alpha,n}^{\pm}(\lambda) = \det\left(T_{J(\lambda)}^{n,\alpha,\pm}\right),$$

then the above identity implies

$$\sum_{\lambda} c_\lambda^P \psi_{\alpha,n}^{\pm}(\lambda + s^n) = \prod_{i=1}^{n} P(x_i)\,x_i^{-d/2}\psi_{\alpha,n}^{\pm}\left(\left(s + \frac{d}{2}\right)^n\right).$$

Finally, put $\alpha = \frac{1}{2}$ (resp., $\alpha = 1$ or $\alpha = 0$) to obtain the formulas for $X = B$ (resp., $X = C$ or $X = D^{\pm}$). Other identities can be proved similarly. ∎

We may regard the following theorem establishing the B, C, and D analogues of Lemma 2.2. We feel that the identities are very interesting.

THEOREM 2.10. *Let $n \in \mathbb{N}$ and let $X = B$, C, or $D^{\pm}$. Let $P(x) = \sum_{i=0}^{d} a_i x^i$ be a polynomial with $\alpha_0 = 1$. Let $s \in \frac{1}{2}\mathbb{N}$ be such that $s \geq d$ if $X = B$ or $D^{\pm}$, and $s \in \mathbb{N}$ such that $s \geq d$ if $X = C$. Then if $P(x)$ is symmetric, we have*

$$\sum_{\substack{\lambda \\ \ell(\lambda) \leq n}} c_{\lambda'}^P t^{|\lambda|}(\lambda + s^n)_{X(n)}$$
$$= \frac{1}{\prod_{i=1}^{n} P(-x_i t)\,P\left(-x_i^{-1} t\right)}$$
$$\times \sum_{\substack{\lambda \\ \ell(\lambda) \leq n}} (-1)^{dn-|\lambda|} c_\lambda^P t^{dn-|\lambda|}\left(\lambda + (s-d)^n\right)_{X(n)},$$

and if $P(x)$ is antisymmetric, we have

$$\sum_{\substack{\lambda \\ \ell(\lambda)\le n}} c_{\lambda'}^{P} t^{|\lambda|} (\lambda + s^n)_{X(n)}$$

$$= \frac{1}{\prod_{i=1}^{n} P(-x_i t) P\left(-x_i^{-1} t\right)}$$

$$\times \sum_{\substack{\lambda \\ \ell(\lambda)\le n}} (-1)^{(d+1)n-|\lambda|} c_{\lambda}^{P} t^{dn-|\lambda|} \left(\lambda + (s-d)^n\right)_{X(n)}.$$

Proof. Let $T^{n,\alpha,\pm}$ be as above and let S_n and S_n' be the n-rowed matrices defined by

$$S_n = \begin{pmatrix} a_0 & -a_1 t^{-1} & a_2 t^{-2} & \cdots & (-1)^d a_d t^{-d} & 0 & \cdots \\ 0 & a_0 & -a_1 t^{-1} & a_2 t^{-2} & \cdots & & \cdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & & a_0 & -a_1 t^{-1} & a_2 t^{-2} & \cdots \end{pmatrix}$$

$$S_n' = \begin{pmatrix} b_0^P & b_1^P t & b_2^P t^2 & b_3^P t^3 & \cdots \\ 0 & b_0^P t & b_1^P t^2 & b_2^P t^3 & \cdots \\ \vdots & \ddots & \ddots & & \\ 0 & \cdots & 0 & b_0^P t^{n-1} & \cdots \end{pmatrix}.$$

The (i,j)-entry of $S_n' \, {}^t T^{n,s+\alpha,\pm}$ is equal to

$$\sum_{k=0}^{\infty} b_k^P t^{k+i-1} \left(x_j^{k+i-1+s+\alpha} \pm x_j^{-k-i+1-s-\alpha} \right)$$

$$= \frac{t^{i-1} x_j^{i-1+s+\alpha}}{P(-t x_j)} \pm \frac{t^{i-1} x^{-i+1-s-\alpha}}{P\left(-t x_j^{-1}\right)}$$

$$= (-1)^d t^{i+d-1} \frac{\left(S_n \, {}^t T^{n,s+\alpha-d,\pm}\right)_{ij}}{P(-t x_j) P\left(-t x_j^{-1}\right)}$$

for $1 \le i, j \le n$. Note that

$$\det(S_n')_{J(\lambda)} = c_{\lambda'}^P t^{|\lambda| + \binom{n}{2}}$$

and

$$\det(S_n)_{J(\lambda)} = (-1)^{|\lambda|} c_{\lambda}^P t^{-|\lambda|}.$$

Let $\psi_{\alpha,n}^{\pm}(\lambda)$ be as in the proof of the preceding theorem. We then use the Cauchy–Binet formula to obtain

$$\begin{aligned}
&\sum_{\lambda} c_{\lambda'}^{P} t^{|\lambda|+\binom{n}{2}} \psi_{\alpha,n}^{\pm}(\lambda + s^n) \\
&\quad = \frac{1}{\prod_{i=1}^{n} P(-tx_j) P\left(-tx_j^{-1}\right)} \\
&\quad\quad \times \sum_{\substack{\lambda \\ \ell(\lambda) \le n}} (-1)^{|\lambda|+dn} t^{-|\lambda|+\binom{n}{2}+dn} c_{\lambda}^{P} \psi_{\alpha,n}^{\pm}\left(\lambda + (s-d)^n\right).
\end{aligned}$$

Finally, put $\alpha = \frac{1}{2}$ (resp., $\alpha = 1$ or $\alpha = 0$) to obtain the formulas for $X = B$ (resp., $X = C$ or $X = D^{\pm}$). The antisymmetric case can be proved in almost the same manner. ∎

3. SCHUR FUNCTION SERIES AND THE ELLIPTIC THETA FUNCTIONS

In this section we investigate the q-specialization of the identities we obtain in the last section. The main point of this section is to show that the infinite product representations of the elliptic theta functions can be proved as a corollary of our formula by employing a purely combinatorial method.

We use the following notation:

$$\begin{aligned}
(a;q)_{\infty} &= \prod_{n=1}^{\infty} (1 - aq^{n-1}), \\
(a;q)_{n} &= \frac{(a;q)_{\infty}}{(aq^n;q)_{\infty}} = \prod_{k=1}^{n} (1 - aq^{k-1}).
\end{aligned} \tag{3.1}$$

The symbols $(a;q)_{\infty}$ and $(a;q)_n$ are abbreviated as $(a)_{\infty}$ and $(a)_n$ when there is no danger of confusion.

LEMMA 3.1. *Let n be a nonnegative integer. Then we have*

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{q^{k(k+n)}}{(q)_k (q)_{k+n}} &= \frac{1}{(q)_{\infty}}, \\
\sum_{k=0}^{\infty} \frac{q^{k(k+n)}}{(q)_k (q)_{k+n+1}} &= \frac{1 + q^{n+1}}{(q)_{\infty}}.
\end{aligned}$$

Proof. Note that $1/(q)_\infty$ is the generating function of all partitions. The first identity can be shown by considering a rectangle like a Durfee square contained in a partition. Let λ be a partition and let r be the maximum integer such that the rectangle of shape $r \times (r+n)$ is contained in λ. We denote this r by $r_n(\lambda)$. Then the generating function of all partitions satisfying $r_n(\lambda) = k$ is given by $q^{k(k+n)}/(q)_k(q)_{k+n}$. Thus, by summing over all k, we obtain the generating function of all partitions. The second identity is derived from the first one as follows:

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{q^{k(k+n)}}{(q)_k(q)_{k+n+1}} &= \sum_{k=0}^{\infty} \frac{q^{k(k+n+1)}}{(q)_k(q)_{k+n+1}} + \sum_{k=1}^{\infty} \frac{q^{k(k+n)}(1-q^k)}{(q)_k(q)_{k+n+1}} \\
&= \frac{1}{(q)_\infty} + q^{n+1} \sum_{k=1}^{\infty} \frac{q^{(k-1)(k+n+1)}}{(q)_{k-1}(q)_{k+n+1}} \\
&= \frac{1+q^{n+1}}{(q)_\infty}.
\end{aligned}$$

This proves the lemma. ∎

COROLLARY 3.2. *Let* $Q_0 = \prod_{n=1}^{\infty}(1-q^{2n})$. *Here* $q = e^{i\tau\pi}(\operatorname{Im}\tau > 0)$. *Then the following hold*:

$$\begin{aligned}
\vartheta_1(v,\tau) &= 2\sum_{k=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)\pi v \\
&= 2q^{1/4} Q_0 \sin \pi v \prod_{n=1}^{\infty} (1 - 2q^{2n}\cos 2\pi v + q^{4n}), \\
\vartheta_2(v,\tau) &= 2\sum_{k=0}^{\infty} q^{(n+1/2)^2} \cos(2n+1)\pi v \\
&= 2q^{1/4} Q_0 \sin \pi v \prod_{n=1}^{\infty} (1 + 2q^{2n}\cos 2\pi v + q^{4n}), \\
\vartheta_3(v,\tau) &= 1 + 2\sum_{k=1}^{\infty} q^{n^2} \cos 2n\pi v \\
&= Q_0 \prod_{n=1}^{\infty} (1 + 2q^{2n-1}\cos 2\pi v + q^{4n-1}), \\
\vartheta_4(v,\tau) &= 1 + 2\sum_{k=1}^{\infty} (-1)^n q^{n^2} \cos 2n\pi v \\
&= Q_0 \prod_{n=1}^{\infty} (1 - 2q^{2n-1}\cos 2\pi v + q^{4n-1}).
\end{aligned}$$

Proof. We prove the identity for ϑ_2. If we consider $n \to \infty$ in the second identity of Proposition 2.8, then we obtain

$$\sum_{n=0}^{\infty} U_{n+1}(a) \sum_{k=0}^{\infty} s_{(2^k 1^n)}(x) = \prod_{n=1}^{\infty} \left(1 + 2ax_n + x_n^2\right).$$

Here $s_{(2^k 1^n)}(x)$ represents the infinite variable Schur function $s_{(2^k 1^n)}(x_1, x_2, \dots)$. Substituting $a = \cos 2\pi v$ into the above identity yields

$$\sum_{n=0}^{\infty} \sin 2(n+1)\pi v \sum_{k=0}^{\infty} s_{(2^k 1^n)}(x) = \sin 2\pi v \prod_{n=1}^{\infty} \left(1 + 2x_n \cos 2\pi v + x_n^2\right), \tag{3.2}$$

owing to the identity $U_{k+1}(\cos 2\pi v) = (\sin 2(n+1)\pi v)/(\sin 2\pi v)$. Furthermore, we substitute $x_n = q^{2n}$ $(n = 1, 2, \dots)$ into this identity, obtaining

$$\begin{aligned}\sum_{n=0}^{\infty} \sin 2(n+1)\pi v \sum_{k=0}^{\infty} q^{2(2k+n)} s_{(2^k 1^n)}(1, q^2, q^4, \dots)& \\ = \sin 2\pi v \prod_{n=1}^{\infty} \left(1 + q^{2n} \cos 2(n+1)\pi v + q^{4n}\right).&\end{aligned}$$

Recall that by the specialization $x_n = q^{n-1}$ of the Schur function $s_\lambda(x)$ with $\lambda = (2^k 1^n)$, we have

$$s_{(2^k 1^n)}(1, q, q^2, \dots) = \frac{q^{n(\lambda)}}{\prod_{\lambda \in \lambda}\left(1 - q^{h(x)}\right)} = \frac{q^{\binom{k}{2} + \binom{k+n}{2}}}{(q)_k (q)_{k+n+1}} \left(1 - q^{n+1}\right), \tag{3.3}$$

where $n(\lambda) = \sum_{i=1}^{\infty}(i-1)\lambda_i$, and $h(x) = \lambda_i + \lambda'_j - i - j + 1$ for $x = (i, j) \in \lambda$. (See [Ma, p. 44, Example 1].) It follows that

$$\begin{aligned}\sum_{k=0}^{\infty} q^{2k+n} s_{(2^k 1^n)}(1, q, q^2, \dots) &= q^{\binom{n+1}{2}}\left(1 - q^{n+1}\right) \sum_{k=0}^{\infty} \frac{q^{k(k+n+1)}}{(q)_k (q)_{k+n+1}} \\ &= \frac{q^{\binom{n+1}{2}}\left(1 - q^{n+1}\right)}{(q)_\infty}.\end{aligned}$$

Combining the above identities, we obtain

$$\sum_{n=0}^{\infty} \sin 2(n+1)\pi v q^{n(n+1)}(1-q^{2n+2})$$
$$= (q^2)_\infty \sin 2\pi v \prod_{n=1}^{\infty} (1 + q^{2n}\cos 2\pi v + q^{4n}).$$

The left-hand side of this identity is equal to

$$\sum_{n=0}^{\infty} \sin 2(n+1)\pi v q^{n(n+1)} - \sum_{n=0}^{\infty} \sin 2(n+1)\pi v q^{(n+1)(n+2)}$$
$$= \sin 2\pi v + \sum_{n=1}^{\infty} q^{n(n+1)}\{\sin 2(n+1)\pi v - \sin 2n\pi v\}$$
$$= 2\sin \pi v \cos \pi v + 2\sum_{n=1}^{\infty} q^{n(n+1)} \cos(2n+1)\pi v \sin \pi v.$$

This proves the identity in question. All other identities can be proved similarly. This completes the proof. ∎

Remark. These theta functions are involved in many fascinating identities of theoretical physics and number theory. In particular, as is widely known, these functions provide an effective way to construct automorphic forms. In fact, A. Weil found that the theta functions are intimately related with a certain projective unitary representation of the symplectic group. From this point of view, to systematize certain aspects of the theory of theta functions, R. Howe introduced the notion of a reductive dual pair of subgroups of the symplectic group [H].

From the viewpoint of representation theory, the infinite product representation formula is obtained by a specialization of the Cauchy Identity, which is a consequence of the Cauchy determinant formula (see the Remark in Section 2 and, e.g., [Wy, p. 202]); in other words, it is related to the restriction of the dual pair $(GL_2(\mathbb{R}), GL_n(\mathbb{R})) \subset Sp_{2n}(\mathbb{R})$ to some smaller pair of groups by letting n go to infinity (see the Remark just below Proposition 2.8). Also, attached to the dual pair $(SL_2(\mathbb{R}), O(n)) \subset Sp_{2n}(\mathbb{R})$ (see [H]), if we denote the quadratic form $x_1^2 + \cdots + x_n^2$ by r^2, the Laplacian $\partial_1^2 + \cdots + \partial_n^2$ by Δ, and the Euler degree operator $x_1\partial_1 + \cdots + x_n\partial_n$ by E, where $\partial_i = \partial/\partial x_i$, the following Capelli-type identity is also known (see, e.g., [Wy, Supplement A], [HU]) as an explicit description of the invariant differential operators in duality:

$$r^2\Delta - E(E+n-2) = \sum_{i<j} (x_i\partial_j - x_j\partial_i)^2.$$

One may say that this is a noncommutative counterpart of the Cauchy−Lagrange Identity. This point was suggested to the second author by T. Umeda.

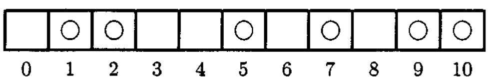
Behind these two seemingly quite different types of formulas, one may naturally find that there is indeed a Cauchy−Binet formula. A part of the primary motivation of this paper is to clarify how the infinite product representation of ϑ_1 (and hence all other ϑ_i) is obtained from the Cauchy−Binet formula by a purely combinatorial method as above.

4. EVALUATION OF SUBDETERMINANTS

In this section we consider the way to calculate all of the subdeterminants of a given rectangular matrix. Here we evaluate minors of a given rectangular matrix, in which the column indices correspond to certain partitions. This evaluation will help us to derive the Littlewood-type formulas in the next section.

First we introduce the notion of Maya diagrams. Fix nonnegative integers $n, N \in \mathbb{N}$ such that $n \leq N$, and let $\mathscr{S}$ denote the interval $[0, N) = \{0, \ldots, N-1\}$. *A Maya diagram* of cardinality n and size N is by definition a subset J of $\mathscr{S}$.

For a Maya diagram J of size N, each point $x \in \mathscr{S}$ is represented by a cell, and a cell x contains a circle if and only if x is in J. For example, if $J = \{1, 2, 5, 7, 9, 10\}$ is a subset of $[0, 11)$, then the Maya diagram that corresponds to J is represented by



In the first section we gave a one-to-one correspondence between an n-element subset J of $\mathscr{S}$ and a partition λ with $\ell(\lambda) \leq n$ and $\lambda_1 \leq N - n$. If a Maya diagram J corresponds to a partition λ by the correspondence, then λ and λ' are obtained from J as follows. Suppose $J = \{a_1 > \cdots > a_n\}$ and assume that the complementary set of J is given by $\mathscr{S} \setminus J = \{b_1 < \cdots < b_{N-n}\}$. Then

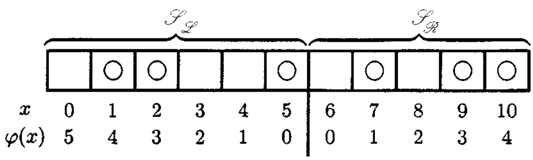
$$\lambda_i = \#\{x \in \mathscr{S} \setminus J;\ x < a_i\}$$
$$\lambda'_i = \#\{x \in J;\ x > b_i\}.$$

For example, the above Maya diagram stands for the partition $\lambda = (554311)$. Let $\mathscr{S}_{\mathscr{L}} = [0, n)$ and $\mathscr{S}_{\mathscr{R}} = [n, N)$ (disjoint subsets of $\mathscr{S}$), and we put $J_{\mathscr{L}} = J \cap \mathscr{S}_{\mathscr{L}}$, $J_{\mathscr{R}} = J \cap \mathscr{S}_{\mathscr{R}}$, $\overline{J_{\mathscr{L}}} = \mathscr{S}_{\mathscr{L}} \setminus J_{\mathscr{L}}$, and $\overline{J_{\mathscr{R}}} = \mathscr{S}_{\mathscr{R}} \setminus J_{\mathscr{R}}$. Then it is easy to see that $p(\lambda) = \#J_{\mathscr{R}} = \#\overline{J_{\mathscr{L}}}$. If we introduce the new coordinates $\varphi(x)$ for each cell $x \in \mathscr{S}$ according to

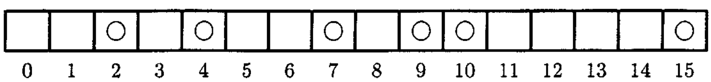
$$\varphi(x) = \begin{cases} n - 1 - x & \text{if } x \in \mathscr{S}_{\mathscr{L}}, \\ x - n & \text{if } x \in \mathscr{S}_{\mathscr{R}}, \end{cases}$$

then the Frobenius notation of λ is given by $(\varphi(J_{\mathscr{R}}) | \varphi(\overline{J_{\mathscr{L}}}))$.

In the above example, $\mathscr{S}_{\mathscr{L}} = [0, 6)$ and $\mathscr{S}_{\mathscr{R}} = [6, 11)$, and we have $J_{\mathscr{R}} = \{7, 9, 10\}$ and $\overline{J_{\mathscr{L}}} = \{0, 3, 4\}$. This implies $p(\lambda) = 3$ and $\lambda = (431|521)$.



Assume that $N \geq 2n$. Let $M = [N/2]$ be the greatest integer that does not exceed $N/2$, and let J be a Maya diagram J of cardinality n and size N. Then J is said to be *balanced* if $x \notin J$ whenever $N - 1 - x \in J$. The following figure gives an example of a balanced Maya diagram $J = \{2, 4, 7, 9, 10, 15\}$ of cardinality $n = 6$ and size $N = 16$.

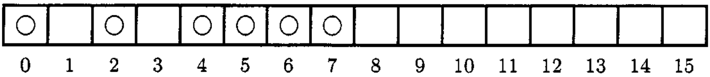


For a balanced Maya diagram, there exists a unique subset $J_0 \subset [0, N)$ that satisfies

(1) $0 \leq j < M$ for all $j \in J_0$,

(2) $j \in J_0$ if and only if $j \in J$ or $N - 1 - j \in J$.

We call this subset J_0 *the fundamental domain* of J, and denote it by $\mathscr{D}(J)$. For the above example of the Maya diagram J, the fundamental domain $\mathscr{D}(J)$ of J is given as follows.

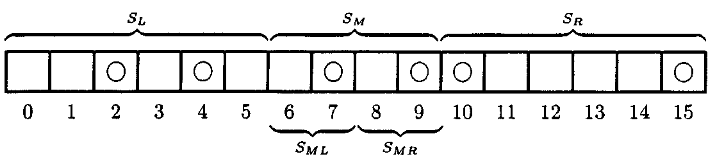


Suppose $a \in J$. From the assumption we have $N - 1 - a \notin J$. Then put $K = (J \setminus \{a\}) \cup \{N - 1 - a\}$. We call the procedure of obtaining K from J *an admissible move* of the circle in the cell a, and call the number of circles between a and $N - 1 - a$ *the leap number*. For a sequence of admissible moves that brings $\mathscr{D}(J)$ to J, let $k(J)$ denote the summation of the leap numbers of these admissible moves. This value $k(J)$ is independent of the sequence we choose and is well defined. We denote $(-1)^{k(J)}$ by sgn J or $(-1)^J$ and call this the sign of J. This quantity will be important for calculating all of the subdeterminants of a given "symmetric matrix." For example, if we move the circle in cell 5 to cell 10 in the above diagram, then the leap number of this admissible move is 2. If we perform the admissible moves on the circles in cells 0, 5 and 6 of $D(J)$, we obtain J, so that $k(J) = 8$ and sgn $J = 1$.

From this point we only consider balanced diagrams. From the condition $N \geq 2n$ we can write $N = 2n + r$ with $r \geq 0$. Let us denote

$$\begin{aligned} S_{\mathrm{L}} &= [0, n), \\ S_{\mathrm{M}} &= [n, n + r), \\ S_{\mathrm{R}} &= [n + r, N), \\ S_{\mathrm{ML}} &= [n, M), \\ S_{\mathrm{MR}} &= [n + r - M, n + r). \end{aligned}$$

We call S_{L}, S_{M}, S_{R}, S_{ML}, and S_{ML} the left, middle, right, middle-left, and middle-right zones of $\mathscr{S}$, respectively. For the above example we have



For a balanced diagram J we put

$$\begin{aligned}
\mathscr{D}_{\mathrm{L}}(J) &= \mathscr{D}(J) \cap S_{\mathrm{L}} \\
\overline{\mathscr{D}}_{\mathrm{L}}(J) &= S_{\mathrm{L}} \setminus \mathscr{D}(J) \\
\mathscr{D}_{\mathrm{M}}(J) &= \mathscr{D}(J) \cap S_{\mathrm{M}} \\
\mathscr{E}_{\mathrm{L}}(J) &= J \cap S_{\mathrm{L}} \\
\mathscr{E}_{\mathrm{R}}(J) &= \mathscr{D}_{\mathrm{L}}(J) \setminus \mathscr{E}_{\mathrm{L}}(J) \\
\mathscr{E}_{\mathrm{ML}}(J) &= J \cap S_{\mathrm{ML}} \\
\mathscr{E}_{\mathrm{MR}}(J) &= \mathscr{D}_{\mathrm{M}}(J) \setminus \mathscr{E}_{\mathrm{ML}}(J) \\
\mathscr{F}_{\mathrm{M}}(J) &= J \cap S_{\mathrm{M}}.
\end{aligned}$$

For the above example $J = \{2, 4, 7, 9, 10, 15\}$, we have $\mathscr{D}_{\mathrm{L}}(J) = \{0, 2, 4, 5\}$, $\overline{\mathscr{D}}_{\mathrm{L}}(J) = \{1, 3\}$, $\mathscr{D}_{\mathrm{M}}(J) = \{6, 7\}$, $\mathscr{E}_{\mathrm{L}}(J) = \{2, 4\}$, $\mathscr{E}_{\mathrm{R}}(J) = \{0, 5\}$, $\mathscr{E}_{\mathrm{ML}}(J) = \{7\}$, $\mathscr{E}_{\mathrm{MR}}(J) = \{6\}$, and $\mathscr{F}_{\mathrm{M}}(J) = \{7, 9\}$.

	S_L						S_M				S_R					
	★		○		○	★	★	○		○	○					○
x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\varphi(x)$	5	4	3	2	1	0	0	1	2	3	4	5	6	7	8	9

For an arbitrary nonnegative integer l, let $\mathscr{I}_l$ denote the set of all l-tuples of the form $\sigma = (\sigma_1, \dots, \sigma_l)$ such that each σ_i is either 0 or 1. For $\sigma \in \mathscr{I}_l$ put $|\sigma| = \sum_{i=1}^{l} \sigma_i$ and $s(\sigma) = \sum_{i=1}^{l} (l-i)\sigma_i$. Furthermore, we write $\mathscr{I}_l^{\pm} = \{\sigma \in \mathscr{I}_l; (-1)^{|\sigma|} = \pm 1\}$. Given $r \in \mathbb{N}$ and $\alpha = (\alpha_1, \dots, \alpha_l) \in [0, r)^l$, we write $t_\sigma^r(\alpha) = (\beta_1, \dots, \beta_l)$, where

$$\beta_i = \begin{cases} \alpha_i & \text{if } \sigma_i = 0, \\ r - 1 - \alpha_i & \text{if } \sigma_i = 1, \end{cases}$$

for each $1 \le i \le l$. For example, if $l = 2$, $\alpha = (0, 1)$ and $r = 4$, then $\mathscr{I}_l = \{(0,0), (0,1), (1,0), (1,1)\}$. Furthermore, if $\sigma = (1, 0)$, then we have $|\sigma| = 1$, $s(\sigma) = 1$, and $t_\sigma^r(\alpha) = (3, 1)$.

Now we are in a position to state our theorem, which gives the partition λ corresponding to J and the sign $(-1)^J$.

THEOREM 4.1. *Let J be a balanced Maya diagram of cardinality n and size $N = 2n + r$. Put $\#\mathscr{F}_{\mathrm{R}}(J) = k$ and $\#\mathscr{E}_{\mathrm{M}}(J) = l$. Furthermore, we put*

$$\varphi\big(\mathscr{E}_{\mathrm{R}}(J)\big) = \{\alpha_1, \dots, \alpha_k\},$$
$$\varphi\big(\mathscr{F}_{\mathrm{M}}(J)\big) = \{\beta_1, \dots, \beta_l\},$$
$$\varphi\big(\overline{\mathscr{D}}_{\mathrm{L}}(J)\big) = \{\gamma_1, \dots, \gamma_l\},$$
$$\varphi\big(\mathscr{D}_{\mathrm{M}}(J)\big) = \{\delta_1 < \cdots < \delta_l\}.$$

Define $\sigma \in \mathscr{I}_l$ on the set $\mathscr{D}_{\mathrm{M}}(J)$ by

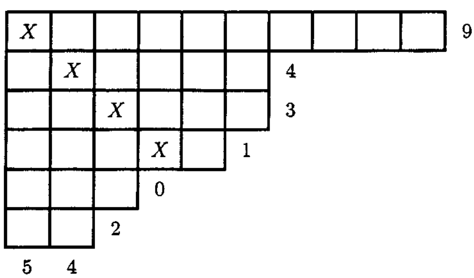
$$\sigma_i = \begin{cases} 0 & \text{if } \delta_i \in \varphi\big(\mathscr{E}_{\mathrm{ML}}(J)\big), \\ 1 & \text{if } \delta_i \in \varphi\big(\mathscr{E}_{\mathrm{MR}}(J)\big), \end{cases}$$

for $1 \le i \le l$. Then we have $p(\lambda) = k + l$, $\beta = \varphi(\mathscr{E}_{\mathrm{ML}}(J)) \cup t_\sigma^r(\varphi(\mathscr{E}_{\mathrm{MR}}(J)))$, and

$$\lambda = (\alpha + r|\alpha) \cup (\beta|\gamma),$$
$$(-1)^J = (-1)^{|\alpha| + \#(\gamma > \alpha) + s(\sigma)}.$$

Here $|\alpha| = \alpha_1 + \cdots + \alpha_k$ and $\#(\gamma > \alpha) = \#\{(i, j);\ \gamma_i > \alpha_j\}$.

Here the α_i are the new coordinates of the circles in the right zone S_{R}, the β_i are those of the circles in the middle zone S_{M}, and the γ_i are those of the empty cells in the left zone S_{L}. For the above example we obtain $k = l = 2$ and $\alpha = \{0, 5\}$, $\beta = \{1, 3\}$, $\gamma = \{2, 4\}$, $\delta = (0, 1)$, $\varphi(\mathscr{E}_{\mathrm{ML}}(J)) = \{1\}$, $\varphi(\mathscr{E}_{\mathrm{MR}}(J)) = \{0\}$, and $\sigma = (1, 0)$. Thus the corresponding partition is given by $(94|50) \cup (31|42) = (9431|5420)$ and is displayed as follows.



Furthermore, in this case, $|\alpha| = 5$ and $\#\{i;\ \beta_i \ge 2\} = 1$ implies that $(-1)^J = +1$.

Proof. Most of our claims are obvious from the definition. The only part we need to prove is the last claim: $(-1)^J = (-1)^{|\alpha| + \#(\gamma > \alpha) + s(\sigma)}$. Suppose $a \in \mathscr{D}(J)$. When $a \in \mathscr{E}_{\mathrm{R}}(J)$, then if $\varphi(a) = \alpha_i$, the leap number of a is equal to $\alpha_i + \#(\gamma > \alpha_i)$. On the other hand, when $a \in \mathscr{E}_{\mathrm{MR}}(J)$, if $\varphi(a) = \delta_i$, it is obvious that the leap number of a is equal to $(l - i)$. This proves our claim regarding $(-1)^J$. ∎

Let B be an n by N matrix with $n \le N$. We assume that the row indices are $1, \ldots, n$, and that the column indices are $0, \ldots, N-1$. We call $B = (\beta_{ij})_{1 \le i \le n,\, 0 \le j < N}$ *column symmetric* if $\beta_{i,j} = \beta_{i,N-1-j}$ for $1 \le i \le n$ and $0 \le j < N$, while we call B *column antisymmetric* if $\beta_{i,j} = -\beta_{i,N-1-j}$ for $1 \le i \le n$ and $0 \le j < N$. The following two lemmas are direct consequences of the definition.

LEMMA 4.2. *Let B be an n by N column symmetric matrix. Fix a subset $J = \{j_1 < \cdots < j_n\}$ of $[0, N)$. Then, unless J contains at most one element of each pair $\{j, N-1-j\}$, the corresponding subdeterminant vanishes.*

LEMMA 4.3. *Let $N = 2n + r$ with $r \ge 0$. Let B be an n by N column symmetric matrix. If J is an n-element balanced subset of $[0, N)$, then* $\det B_J = (-1)^J \det B_{D(J)}$.

COROLLARY 4.4. *Let $N = 2n + r$ with $r \ge 0$. Let s be the greatest integer that does not exceed $r/2$. Let B be an n by N matrix.*

(1) *Assume that B is column symmetric. Then we have*

$$\det B_{J(\lambda)} = (-1)^{|\alpha| + \#(\gamma > \alpha) + s(\sigma)} \det B_{J((\delta|\gamma))}$$

if λ is in the form $\lambda = (\alpha + r|\alpha) \cup (t_\sigma^r(\delta)|\gamma)$ for some $\alpha, \gamma \in [0, n)$, $\delta \in [0, s)$ and $\sigma \in \mathscr{I}_l$ with $\#\gamma = \#\delta = l \ge 0$. Otherwise $\det B_{J(\lambda)}$ *vanishes.*

(2) *Assume that B is column antisymmetric. Then we have*

$$\det B_{J(\lambda)} = (-1)^{|\alpha| + \#(\gamma > \alpha) + s(\sigma) + |\sigma|} \det B_{J((\delta|\gamma))}$$

if λ is in the form $\lambda = (\alpha + r|\alpha) \cup (t_\sigma^r(\delta)|\gamma)$ for some $\alpha, \gamma \in [0, n)$, $\delta \in [0, s)$, and $\sigma \in \mathscr{I}_l$ with $\#\gamma = \#\delta = l \ge 0$. Otherwise $\det B_{J(\lambda)}$ *vanishes.*

5. LITTLEWOOD-TYPE FORMULAS

The famous identities of Schur functions found by Littlewood are those given at the beginning of Section 0. We give these identities again below because of their importance. The reader can find remarkable proofs of these identities using the Root system in Example 9 of Section 5 in [Mc] (also see [IOW]). We have

$$\sum_{\lambda=(\alpha-1|\alpha)} (-1)^{|\lambda|/2} s_\lambda(x) = \prod_{i<j}(1-x_i x_j)$$

$$\sum_{\lambda=(\alpha|\alpha)} (-1)^{(|\lambda|+p(\lambda))/2} s_\lambda(x) = \prod_{i}(1-x_i)\prod_{i<j}(1-x_i x_j)$$

$$\sum_{\lambda=(\alpha+1|\alpha)} (-1)^{|\lambda|/2} s_\lambda(x) = \prod_{i}(1-x_i^2)\prod_{i<j}(1-x_i x_j).$$

The identities we consider in this section are generalizations of these identities. We give a general formula that gives the expansion of the product $\prod_i P(x_i) \prod_{i<j}(1-x_i x_j)$ into a linear combination of Schur functions when we are given a symmetric or antisymmetric polynomial $P(x)$.

The Schur functions are known to be the characters of the general linear groups. This means that these identities give character formulas on the classical groups of type A.

The B, C, and D analogues of these identities are given in the next section. The main tools we use are the Cauchy–Binet formula and the theorem on subdeterminants we obtained above. Some of these identities are generalizations of the results in [LP] and [YW].

We gave a proof of the above classical Littlewood formulas using the minor summation formula of Pfaffians in [IOW]. The following theorem gives a generalization of the classical Littlewood formulas, and the proof that follows gives an improved and simplified version of our proof given in [IOW]. In particular we found that this type of identity can be proved by the Binet–Cauchy formula instead of the minor summation formula of Pfaffians.

THEOREM 5.1. *Let n and r be nonnegative integers, and let $P(x) = \sum_{i=0}^{d} a_i x^i$ be a polynomial of degree d with $a_0 = 1$. If $P(x)$ is symmetric, then we have*

$$\begin{aligned}\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\emptyset}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{S}_\nu} &(-1)^{|\alpha|+\#(\alpha<\gamma)+s(\sigma)} \\ &\times c^P_{(\beta|\gamma)} s_{(\alpha+r+d|\alpha)\cup(t_\sigma^{r+d}(\beta)|\gamma)}(x) \\ &= 2\prod_{i=1}^{n} P(x_i) \prod_{1\le i<j\le n}(1-x_i x_j) \\ &\quad\times \prod_{i=1}^{n} x_i^{(r+1)/2} \left[\frac{r+1}{2},\dots,\frac{r+1}{2}\right]^{+}_{D(n)},\end{aligned} \tag{5.1}$$

and

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\varnothing}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu} (-1)^{|\alpha|+\#\alpha+\#(\alpha<\gamma)+s(\sigma)+|\sigma|}$$

$$\times c^P_{(\beta|\gamma)} s_{(\alpha+r+d|\alpha)\cup(t_\sigma^{r+d}(\beta)|\gamma)}(x)$$

$$= \prod_{i=1}^{n} P(x_i)(1-x_i) \prod_{1\le i<j\le n} (1-x_ix_j) \prod_{i=1}^{n} x_i^{r/2} \left(\frac{r}{2},\dots,\frac{r}{2}\right)_{B(n)}$$

$$= \prod_{i=1}^{n} P(x_i)\big(1-x_i^2\big) \prod_{1\le i<j\le n} (1-x_ix_j)$$

$$\times \prod_{i=1}^{n} x_i^{(r-1)/2} \left(\frac{r-1}{2},\dots,\frac{r-1}{2}\right)_{C(n)}$$

$$= 2(-1)^n \prod_{i=1}^{n} P(x_i) \prod_{1\le i<j\le n} (1-x_ix_j)$$

$$\times \prod_{i=1}^{n} x_i^{(r+1)/2} \left[\frac{r+1}{2},\dots,\frac{r+1}{2}\right]^{-}_{D(n)}. \tag{5.2}$$

From the above two identities, we obtain

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\alpha\equiv 0\ (\mathrm{mod}\ 2)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\varnothing}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu^{\pm}} (-1)^{|\alpha|+\#(\alpha<\gamma)+s(\sigma)}$$

$$\times c^P_{(\beta|\gamma)} s_{(\alpha+r+d|\alpha)\cup(t_\sigma^{r+d}(\beta)|\gamma)}(x)$$

$$+ \sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\alpha\not\equiv 0\ (\mathrm{mod}\ 2)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\varnothing}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu^{\mp}} (-1)^{|\alpha|+\#(\alpha<\gamma)+s(\sigma)}$$

$$\times c^P_{(\beta|\gamma)} s_{(\alpha+r+d|\alpha)\cup(t_\sigma^{r+d}(\beta)|\gamma)}(x)$$

$$= \prod_{i=1}^{n} P(x_i) \prod_{1\le i<j\le n} (1-x_ix_j) \prod_{i=1}^{n} x_i^{(r+1)/2} \left(\frac{r+1}{2},\dots,\frac{r+1}{2}\right)^{\pm}_{D(n)}. \tag{5.3}$$

If $P(x)$ is antisymmetric, then we have

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\emptyset}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu} (-1)^{|\alpha|+\#\alpha+\#(\alpha<\gamma)+s(\sigma)+|\sigma|} \times c^P_{(\beta|\gamma)} s_{(\alpha+r+d|\alpha)\cup(t^{r+d}_\sigma(\beta)|\gamma)}(x)$$
$$= 2\prod_{i=1}^{n} P(x_i) \prod_{1\le i<j\le n} (1-x_ix_j) \times \prod_{i=1}^{n} x_i^{(r+1)/2} \left[\frac{r+1}{2},\dots,\frac{r+1}{2}\right]^+_{D(n)}, \tag{5.4}$$

and

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\emptyset}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu} (-1)^{|\alpha|+\#(\alpha<\gamma)+s(\sigma)} \times c^P_{(\beta|\gamma)} s_{(\alpha+r+d|\alpha)\cup(t^{r+d}_\sigma(\beta)|\gamma)}(x)$$
$$= \prod_{i=1}^{n} P(x_i)(1-x_i) \prod_{1\le i<j\le n} (1-x_ix_j) \prod_{i=1}^{n} x_i^{r/2} \left(\frac{r}{2},\dots,\frac{r}{2}\right)_{B(n)}$$
$$= \prod_{i=1}^{n} P(x_i)\left(1-x_i^2\right) \prod_{1\le i<j\le n} (1-x_ix_j) \times \prod_{i=1}^{n} x_i^{(r-1)/2} \left(\frac{r-1}{2},\dots,\frac{r-1}{2}\right)_{C(n)}$$
$$= 2(-1)^n \prod_{i=1}^{n} P(x_i) \prod_{1\le i<j\le n} (1-x_ix_j) \times \prod_{i=1}^{n} x_i^{(r+1)/2} \left[\frac{r+1}{2},\dots,\frac{r+1}{2}\right]^-_{D(n)}. \tag{5.5}$$

From the above two identities, we obtain

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\alpha\equiv 0\ (\mathrm{mod}\ 2)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\emptyset}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}^{\pm}_\nu} (-1)^{|\alpha|+\#\alpha+\#(\alpha<\gamma)+s(\sigma)+|\sigma|} \times c^P_{(\beta|\gamma)} s_{(\alpha+r+d|\alpha)\cup(t^{r+d}_\sigma(\beta)|\gamma)}(x)$$

$$+\sum_{\nu=0}^{d}\sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\alpha\not\equiv 0\ (\mathrm{mod}\ 2)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\varnothing}}\sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}}\sum_{\sigma\in\mathscr{I}_\nu^{\mp}}(-1)^{|\alpha|+\#\alpha+\#(\alpha<\gamma)+s(\sigma)+|\sigma|}$$

$$\times c^{P}_{(\beta|\gamma)}s_{(\alpha+r+d|\alpha)\cup(t_\sigma^{r+d}(\beta)|\gamma)}(x)$$

$$=\prod_{i=1}^{n}P(x_i)\prod_{1\le i<j\le n}(1-x_ix_j)\prod_{i=1}^{n}x_i^{(r+1)/2}\left(\frac{r+1}{2},\ldots,\frac{r+1}{2}\right)^{\pm}_{D(n)}. \tag{5.6}$$

Before we proceed to the proof of this theorem, we give some examples of these identities. For convention, we use the following notation:

$$f_r^{B(n)}(x)=\prod_i(1-x_i)\prod_{i<j}(1-x_ix_j)\prod_i x_i^{r/2}\left(\frac{r}{2},\ldots,\frac{r}{2}\right)_{B(n)}$$

$$f_r^{C(n)}(x)=\prod_i(1-x_i^2)\prod_{i<j}(1-x_ix_j)\prod_i x_i^{(r-1)/2}\left(\frac{r-1}{2},\ldots,\frac{r-1}{2}\right)_{C(n)}$$

$$F_r^{D^{\pm}(n)}(x)=2\prod_{i<j}(1-x_ix_j)\prod_i x_i^{(r+1)/2}\left[\frac{r+1}{2},\ldots,\frac{r+1}{2}\right]^{\pm}_{D(n)},$$

and $2f_r^{D^{\pm}(n)}(x)=F_r^{D^{+}(n)}(x)\pm F_r^{D^{-}(n)}(x)$.

EXAMPLES. Let $|\alpha|'=|\alpha|+\#\alpha$. If we put $P(x)=1-x$ in (5.4) then we have

$$\sum_{\lambda=(\alpha+r+1|\alpha)}(-1)^{|\alpha|'}s_\lambda$$

$$+\sum_k\sum_{\substack{\lambda=(\alpha+r+1|\alpha)\\ \alpha\not\ni k}}(-1)^{|\alpha|'+\#(\alpha<k)+k+1}\{s_{\lambda\cup(0|k)}-s_{\lambda\cup(r|k)}\}$$

$$=\prod_i(1-x_i)F_r^{D^{+}(n)}(x) \tag{5.7}$$

and, if we put $P(x)=1-x$ in (5.5) then we have

$$\sum_{\lambda=(\alpha+r+1|\alpha)}(-1)^{|\alpha|}s_\lambda$$

$$+\sum_k\sum_{\substack{\lambda=(\alpha+r+1|\alpha)\\ \alpha\not\ni k}}(-1)^{|\alpha|+\#(\alpha<k)+k+1}\{s_{\lambda\cup(0|k)}+s_{\lambda\cup(r|k)}\}$$

$$= \prod_i (1 - x_i) f_r^{B(n)}(x) = \prod_i (1 - x_i) f_r^{C(n)}(x)$$

$$= (-1)^n \prod_i (1 - x_i) F_r^{D^-(n)}(x). \tag{5.8}$$

From the above two identities, we obtain the following formula for the characters of $D(n)$:

$$\sum_{\substack{\lambda=(\alpha+r+1|\alpha)\\ \alpha\in\Gamma_n^{\pm}}} (-1)^{|\alpha|'} s_\lambda + \sum_k \sum_{\substack{\lambda=(\alpha+r+1|\alpha)\\ \alpha\not\ni k\\ \alpha\in\Gamma_n^{\pm}}} (-1)^{|\alpha|'+\#(\alpha<k)+k+1} s_{\lambda\cup(0|k)}$$

$$+ \sum_k \sum_{\substack{\lambda=(\alpha+r+1|\alpha)\\ \alpha\not\ni k\\ \alpha\in\Gamma_n^{\mp}}} (-1)^{|\alpha|'+\#(\alpha<k)+k} s_{\lambda\cup(r|k)}$$

$$= \prod_i (1 - x_i) f_r^{D^{\pm}(n)}(x). \tag{5.9}$$

Here $\Gamma_n^{\pm} = \{d \in [0,n)\colon (-1)^{\#\alpha+n} = \pm 1\}$. If we put $r = 0$ in (5.8), we obtain the following identity, which can also be derived from Theorem 5.1:

$$\sum_{\lambda=(\alpha+1|\alpha)} (-1)^{|\alpha|} s_\lambda + 2\sum_k \sum_{\substack{\lambda=(\alpha+1|\alpha)\\ \alpha\not\ni k}} (-1)^{|\alpha|+\#(\alpha<k)+k+1} s_{\lambda\cup(0|k)}$$

$$= \prod_i (1 - x_i)^2 \prod_{i<j} (1 - x_i x_j). \tag{5.10}$$

Some identities of this type are established in [IW1].

Next we state the $P(x) = 1 + 2ax + x^2$ case as an example of the case when $P(x)$ is symmetric in Theorem 5.1. If we substitute the results obtained in Section 2 into (5.1) and (5.2), then we have

$$\sum_{\lambda=(\alpha+r+2|\alpha)} (-1)^{|\alpha|} s_\lambda$$

$$+ \sum_k \sum_{\substack{\lambda=(\alpha+r+2|\alpha)\\ \alpha\not\ni k}} (-1)^{|\alpha|+\#(\alpha<k)} U_{k+2}(a)\{s_{\lambda\cup(0|k)} + s_{\lambda\cup(r+1|k)}\}$$

$$+ \sum_k \sum_{\substack{\lambda=(\alpha+r+2|\alpha)\\ \alpha\not\ni k}} (-1)^{|\alpha|+\#(\alpha<k)} U_{k+1}(a)\{s_{\lambda\cup(1|k)} + s_{\lambda\cup(r|k)}\}$$

$$+ \sum_{k<l} \sum_{\substack{\lambda=(\alpha+r+2|\alpha)\\ \alpha\not\ni k,l}} (-1)^{|\alpha|+\#(\alpha<k)+\#(\alpha<l)} U_{l-k+1}(a)$$

$$\times\{s_{\lambda\cup(1,0|kl)} - s_{\lambda\cup(r+1,1|kl)} + s_{\lambda\cup(r,0|kl)} - s_{\lambda\cup(r+1,r|kl)}\}$$

$$= \prod_i (1 + 2ax + x_i^2) F_r^{D^+(n)}(x) \tag{5.11}$$

and

$$\sum_{\lambda=(\alpha+r+2|\alpha)} (-1)^{|\alpha|'} s_\lambda + \sum_k \sum_{\substack{\lambda=(\alpha+r+2|\alpha)\\ \alpha\not\ni k}} (-1)^{|\alpha|'+\#(\alpha<k)}$$

$$\times U_{k+2}(a)\{s_{\lambda\cup(0|k)} - s_{\lambda\cup(r+1|k)}\}$$

$$+ \sum_k \sum_{\substack{\lambda=(\alpha+r+2|\alpha)\\ \alpha\not\ni k}} (-1)^{|\alpha|'+\#(\alpha<k)} U_{k+1}(a)\{s_{\lambda\cup(1|k)} - s_{\lambda\cup(r|k)}\}$$

$$+ \sum_{k<l} \sum_{\substack{\lambda=(\alpha+r+2|\alpha)\\ \alpha\not\ni k,l}} (-1)^{|\alpha|'+\#(\alpha<k)+\#(\alpha<l)} U_{l-k+1}(a)$$

$$\times\{s_{\lambda\cup(1,0|kl)} + s_{\lambda\cup(r+1,1|kl)} - s_{\lambda\cup(r,0|kl)} - s_{\lambda\cup(r+1,r|kl)}\}$$

$$= \prod_i (1 + 2ax_i + x_i^2) f_r^{B(n)}(x) = \prod_i (1 + 2ax_i + x_i^2) f_r^{C(n)}(x)$$

$$= (-1)^n \prod_i (1 + 2ax_i + x_i^2) F_r^{D^-(n)}(x). \tag{5.12}$$

Similarly we obtain the case $P(x) = 1 - x^2$ from (5.4) and (5.5):

$$\sum_{\lambda=(\alpha+r+2|\alpha)} (-1)^{|\alpha|'} s_\lambda$$

$$+ \sum_{k:\,\text{odd}} \sum_{\substack{\lambda=(\alpha+r+2|\alpha)\\ \alpha\not\ni k}} (-1)^{|\alpha|'+\#(\alpha<k)}\{s_{\lambda\cup(0|k)} - s_{\lambda\cup(r+1|k)}\}$$

$$+ \sum_{k:\,\text{even}} \sum_{\substack{\lambda=(\alpha+r+2|\alpha)\\ \alpha\not\ni k}} (-1)^{|\alpha|'+\#(\alpha<k)}\{s_{\lambda\cup(1|k)} - s_{\lambda\cup(r|k)}\}$$

$$+ \sum_{\substack{k<l\\ l-k:\,\text{odd}}} \sum_{\substack{\lambda=(\alpha+r+2|\alpha)\\ \alpha\not\ni k,l}} (-1)^{|\alpha|'+\#(\alpha<k)+\#(\alpha<l)+k}$$

$$\times\{s_{\lambda\cup(1,0|kl)} + s_{\lambda\cup(r+1,1|kl)} - s_{\lambda\cup(r,0|Kl)} - s_{\lambda\cup(r+1,r|kl)}\}$$

$$= \prod_i (1 - x_i^2) F_r^{D^+(n)}(x), \tag{5.13}$$

and

$$\sum_{\lambda=(\alpha+r+2|\alpha)} (-1)^{|\alpha|} s_\lambda$$

$$+ \sum_{k:\,\text{odd}} \sum_{\substack{\lambda=(\alpha+r+2|\alpha)\\ \alpha\not\ni k}} (-1)^{|\alpha|+\#(\alpha<k)}\{s_{\lambda\cup(0|k)} + s_{\lambda\cup(r+1|k)}\}$$

$$+ \sum_{k:\,\text{even}} \sum_{\substack{\lambda=(\alpha+r+2|\alpha)\\ \alpha\not\ni k}} (-1)^{|\alpha|+\#(\alpha<k)}\{s_{\lambda\cup(1|k)} + s_{\lambda\cup(r|k)}\}$$

$$+ \sum_{\substack{k<l \\ l-k:\,\mathrm{odd}}} \sum_{\substack{\lambda=(\alpha+r+2|\alpha) \\ \alpha\not\ni k,l}} (-1)^{|\alpha|+\#(\alpha<k)+\#(\alpha<l)+k}$$
$$\times \{ s_{\lambda\cup(1,0|kl)} - s_{\lambda\cup(r+1,1|kl)} + s_{\lambda\cup(r,0|kl)} - s_{\lambda\cup(r+1,r|kl)} \}$$
$$= \prod_i (1-x_i^2) f_r^{B(n)}(x) = \prod_i (1-x_i^2) f_r^{C(n)}(x)$$
$$= (-1)^n \prod_i (1-x_i^2) F_r^{D^-(n)}(x). \tag{5.14}$$

In particular, if we put $r = 1$ in (5.12), we obtain

$$\sum_{\lambda=(\alpha+3|\alpha)} (-1)^{|\alpha|'} s_\lambda$$
$$+ \sum_k \sum_{\substack{\lambda=(\alpha+3|\alpha) \\ \alpha\not\ni k}} (-1)^{|\alpha|+\#(\alpha<k)} U_{k+2}(a) \{ s_{\lambda\cup(0|k)} - s_{\lambda\cup(2|k)} \}$$
$$= \prod_i (1+2ax_i+x_i^2) \prod_{i\le j} (1-x_ix_j), \tag{5.15}$$

and, if we put $r = 1$ in (5.14), then we obtain

$$\sum_{\lambda=(\alpha+3|\alpha)} (-1)^{|\alpha|} s_\lambda + \sum_{k:\,\mathrm{odd}} \sum_{\substack{\lambda=(\alpha+3|\alpha) \\ \alpha\not\ni k}} (-1)^{|\alpha|+\#(\alpha<k)} \{ s_{\lambda\cup(0|k)} + s_{\lambda\cup(2|k)} \}$$
$$+2 \sum_{k:\,\mathrm{even}} \sum_{\substack{\lambda=(\alpha+3|\alpha) \\ \alpha\not\ni k}} (-1)^{|\alpha|+\#(\alpha<k)} s_{\lambda\cup(1|k)}$$
$$+2 \sum_{\substack{k<l \\ k\equiv l\ (\mathrm{mod}\ 2)}} \sum_{\substack{\lambda=(\alpha+3|\alpha) \\ \alpha\not\ni k,l}} (-1)^{|\alpha|+\#(\alpha<k)+\#(\alpha<l)+k} \{ s_{\lambda\cup(1,0|kl)} - s_{\lambda\cup(2,1|kl)} \}$$
$$= \prod_i (1-x_i^2) \prod_{i\le j} (1-x_ix_j). \tag{5.16}$$

For lack of space, we refrain from displaying more examples. Let us simply note that we can derive far more examples of the kinds of identities that are given in [IW1].

We now state the proof of Theorem 5.1.

Proof. We define the n by $2n+r$ matrix $V^{(n,r,\pm)} = (v_{ik}^{(n,r,\pm)})$ by

$$v_{ik}^{(n,r,\pm)} = \begin{cases} 1 & \text{if } k = i \text{ for } 1 \le i \le n, \\ \pm 1 & \text{if } k = 2n+r+1-i \text{ for } 1 \le i \le n, \\ 0 & \text{otherwise}, \end{cases}$$

for $r \geq 0$. For example, $V^{(n,0,-)}$ assumes the following form:

$$V^{(0,-)} = \begin{pmatrix} 1 & 0 & & & \cdots & \cdots & & & 0 & -1 \\ 0 & 1 & & & & & & & -1 & 0 \\ \vdots & & \ddots & & & & & \iddots & & \vdots \\ 0 & \cdots & & 1 & 0 & 0 & -1 & & \cdots & 0 \\ 0 & \cdots & & 0 & 1 & -1 & 0 & & \cdots & 0 \end{pmatrix}.$$

In [IOW], we discussed the determinant $\det V^{(n,r,\pm)}\,{}^tT^{(n,2n+r)}$. In the case of $V^{(n,r,+)}$, $\det V^{(n,r,+)}\,{}^tT^{(n,2n+r)}$ is given by

$$\begin{aligned} &\begin{vmatrix} 1 + x_1^{2n+r-1} & x_1 + x_1^{2n+r-2} & \cdots & x_1^{n-1} + x_1^{n+r} \\ \vdots & \vdots & \ddots & \vdots \\ 1 + x_n^{2n+r-1} & x_n + x_n^{2n+r-2} & \cdots & x_n^{n-1} + x_n^{n+r} \end{vmatrix} \\ &\quad = \prod_{i=1}^{n} x_i^{(2n+r-1)/2} \\ &\quad\quad \times \begin{vmatrix} x_1^{-(2n+r-1)/2} + x_1^{(2n+r-1)/2} & x_1^{-(2n+r-3)/2} + x_1^{(2n+r-3)/2} & \cdots & x_1^{-(r+1)/2} + x_1^{(r+1)/2} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{-(2n+r-1)/2} + x_n^{(2n+r-1)/2} & x_n^{-(2n+r-3)/2} + x_n^{(2n+r-3)/2} & \cdots & x_n^{-(r+1)/2} + x_n^{(r+1)/2} \end{vmatrix} \\ &\quad = \prod_{i<j} (x_j - x_i)(1 - x_i x_j) \prod_i x_i^{(r+1)/2} \left[\frac{r+1}{2}, \ldots, \frac{r+1}{2} \right]^{+}_{D(n)} \\ &\quad = F_r^{D^+(n)}(x). \end{aligned}$$

Similarly, in the case of $V^{(n,r,-)}$, $\det V^{(n,r,-)}\,{}^tT^{(n,2n+r)}$ becomes

$$f_r^{B(n)}(x) = f_r^{C(n)}(x) = (-1)^n F_r^{D^-(n)}(x),$$

and, in the case of $V^{(n)}$, $\det V^{(n)}\,{}^tT^{(n,2n-1)} = \prod_{i<j}(x_j - x_i)(1 - x_i x_j)$. Next we apply the Cauchy–Binet formula to the matrix of the form $B = V^{(n,r,\pm)} M^{P,n+2r}$ (see the beginning of Section 2). We thus obtain

$$\begin{aligned} \det B\,{}^tT^{(n,2n+r+d)} &= \det V^{(n,r,\pm)} M^{P,n+2r}\,{}^tT^{(n,2n+r+d)} \\ &= \det V^{(n,r,\pm)}\,{}^tT^{(n,2n+r)} \operatorname{diag}(P(x_1), \ldots, P(x_n)) \\ &= \prod_{i=1}^{n} P(x_i) \cdot \det V^{(n,r,\pm)}\,{}^tT^{(n,2n+r)}. \end{aligned}$$

This implies that

$$\sum_{\substack{I \subset [N] \\ \# I = n}} \det\left(V^{(n,r,+)} M^{P,n+2r}\right)_I \det T_I^{(n,2n+r+d)} = \prod_{i=1}^{n} P(x_i) F_r^{D^+(n)}(x),$$

and

$$\sum_{\substack{I \subset [N] \\ \# I = n}} \det\left(V^{(n,r,+)} M^{P,n+2r}\right)_I \det T_I^{(n,2n+r+d)}$$
$$= \prod_{i=1}^{n} P(x_i) f_r^{B(n)}(x) = \prod_{i=1}^{n} P(x_i) f_r^{C(n)}(x)$$
$$= \prod_{i=1}^{n} P(x_i) F_r^{D^-(n)}(x).$$

Using this identity and Theorem 4.1, we obtain the theorems. ∎

6. LITTLEWOOD-TYPE FORMULAS OF B, C, AND D TYPES

In this section we use the methods of the previous section and give analogues for summations of characters of the other classical groups. Theorems 6.1, 6.2, and 6.3 in this section give the B, C, and D analogues of Theorem 5.1, respectively.

THEOREM 6.1. *Let $n, r \in \mathbb{N}$ and $s \in \frac{1}{2}\mathbb{N}$, and let $P(x) = \sum_{i=0}^{d} a_i x^i$ be a polynomial of degree d with $a_0 = 1$. If $P(x)$ is symmetric, then we have*

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha, \gamma \subset [0,n) \\ \#\gamma = \nu \\ \alpha \cap \gamma = \emptyset}} \sum_{\substack{\beta \subset [0,d) \\ \#\beta = \nu}} \sum_{\sigma \in \mathscr{I}_\nu} (-1)^{|\alpha| + \#(\alpha < \gamma) + s(\sigma)}$$
$$\times c^P_{(\beta|\gamma)} \left((\alpha + r + d | \alpha) \cup \left(t_\sigma^{r+d}(\beta) | \gamma \right) + s^n \right)_{B(n)}$$
$$= (-1)^{n(n-1)/2} \prod_{i=1}^{n} P(x_i) x_i^{-d/2}$$
$$\times \frac{x_i^{n+s+(r+d)/2} - x^{-n-s-(r+d)/2}}{x_i^{1/2} - x_i^{-1/2}} \left[\left(\frac{r+1}{2} \right)^n \right]^+_{D(n)}, \quad (6.1)$$

and

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\varnothing}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu} (-1)^{|\alpha|+\#\alpha+\#(\alpha<\gamma)+s(\sigma)+|\sigma|}$$

$$\times c^{P}_{(\beta|\gamma)}\Big((\alpha+r+d|\alpha)\cup\big(t_\sigma^{r+d}(\beta)|\gamma\big)\Big)_{B(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)\,x_i^{-d/2}$$

$$\times\Big(x_i^{n+s+(r+d)/2}+x_i^{-n-s-(r+d)/2}\Big)\left(\left(\frac{r}{2}\right)^n\right)_{B(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)\,x_i^{-d/2}$$

$$\times\Big(x_i^{n+s+(r+d)/2}+x_i^{-n-s-(r+d)/2}\Big)\Big(x_i^{1/2}+x_i^{-1/2}\Big)\left(\left(\frac{r-1}{2}\right)^n\right)_{C(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)\,x_i^{-d/2}$$

$$\times\frac{x_i^{n+s+(r+d)/2}+x_i^{-n-s-(r+d)/2}}{x_i^{1/2}-x_i^{-1/2}}\left[\left(\frac{r+1}{2}\right)^n\right]^{-}_{D(n)}. \tag{6.2}$$

If $P(x)$ is antisymmetric, then we have

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\varnothing}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu} (-1)^{|\alpha|+\#\alpha+\#(\alpha<\gamma)+s(\sigma)+|\sigma|}$$

$$\times c^{P}_{(\beta|\gamma)}\Big((\alpha+r+d|\alpha)\cup\big(t_\sigma^{r+d}(\beta)|\gamma\big)+s^n\Big)_{B(n)}$$

$$= (-1)^{n(n-1)/2}\prod_{i=1}^{n} P(x_i)\,x_i^{-d/2}$$

$$\times\frac{x_i^{n+s+(r+d)/2}+x_i^{-n-s-(r+d)/2}}{x_i^{1/2}-x_i^{-1/2}}\left[\left(\frac{r+1}{2}\right)^n\right]^{+}_{D(n)}, \tag{6.3}$$

and

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\emptyset}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu} (-1)^{|\alpha|+\#(\alpha<\gamma)+s(\sigma)}$$

$$\times c^P_{(\beta|\gamma)}\Big((\alpha+r+d|\alpha)\cup\big(t^{r+d}_\sigma(\beta)|\gamma\big)\Big)_{B(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)x_i^{-d/2}$$

$$\times\big(x_i^{n+s+(r+d)/2}-x_i^{-n-s-(r+d)/2}\big)\left(\left(\frac{r}{2}\right)^n\right)_{B(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)x_i^{-d/2}$$

$$\times\big(x_i^{n+s+(r+d)/2}-x_i^{-n-s-(r+d)/2}\big)$$

$$\times\big(x_i^{1/2}+x_i^{-1/2}\big)\left(\left(\frac{r-1}{2}\right)^n\right)_{C(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)x_i^{-d/2}$$

$$\times\frac{x_i^{n+s+(r+d)/2}-x_i^{-n-s-(r+d)/2}}{x_i^{1/2}-x_i^{-1/2}}\left[\left(\frac{r+1}{2}\right)^n\right]^-_{D(n)}. \quad (6.4)$$

THEOREM 6.2. *Let $n, r\in\mathbb{N}$ and $s\in\frac{1}{2}\mathbb{N}$, and let $P(x)=\sum_{i=0}^{d}a_ix^i$ be a polynomial of degree d with $a_0=1$. If $P(x)$ is symmetric, then we have*

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\emptyset}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu} (-1)^{|\alpha|+\#(\alpha<\gamma)+s(\sigma)}$$

$$\times c^P_{(\beta|\gamma)}\Big((\alpha+r+d|\alpha)\cup\big(t^{r+d}_\sigma(\beta)|\gamma\big)+s^n\Big)_{C(n)}$$

$$= (-1)^{n(n-1)/2}\prod_{i=1}^{n} P(x_i)x_i^{-d/2}$$

$$\times\frac{x_i^{n+s+(r+d+1)/2}-x_i^{-n-s-(r+d+1)/2}}{x_i^{1/2}-x_i^{-1/2}}\left[\left(\frac{r+1}{2}\right)^n\right]^+_{D(n)}, \quad (6.5)$$

and

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\varnothing}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu} (-1)^{|\alpha|+\#\alpha+\#(\alpha<\gamma)+s(\sigma)+|\sigma|}$$

$$\times c^{P}_{(\beta|\gamma)}\Big((\alpha+r+d|\alpha)\cup\big(t^{r+d}_{\sigma}(\beta)|\gamma\big)\Big)_{C(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)x_i^{-d/2}$$

$$\times\left(x_i^{n+s+(r+d+1)/2}+x_i^{-n-2-(r+d+1)/2}\right)\left(\left(\frac{r}{2}\right)^{n}\right)_{B(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)x_i^{-d/2}$$

$$\times\left(x_i^{n+s+(r+d+1)/2}+x_i^{-n-2-(r+d+1)/2}\right)$$

$$\times\left(x_i^{1/2}+x_i^{-1/2}\right)\left(\left(\frac{r-1}{2}\right)^{n}\right)_{C(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)x_i^{-d/2}$$

$$\times\frac{x_i^{n+s+(r+d+1)/2}+x_i^{-n-s-(r+d+1)/2}}{x_i^{1/2}-x_i^{-1/2}}\left[\left(\frac{r+1}{2}\right)^{n}\right]^{-}_{D(n)}. \quad (6.6)$$

If $P(x)$ is antisymmetric, then we have

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\varnothing}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu} (-1)^{|\alpha|+\#\alpha+\#(\alpha<\gamma)+s(\sigma)+|\sigma|}$$

$$\times c^{P}_{(\beta|\gamma)}\Big((\alpha+r+d|\alpha)\cup\big(t^{r+d}_{\sigma}(\beta)|\gamma\big)+s^{n}\Big)_{C(n)}$$

$$= (-1)^{n(n-1)/2}\prod_{i=1}^{n} P(x_i)x_i^{-d/2}$$

$$\times\frac{x_i^{n+s+(r+d+1)/2}+x_i^{-n-s-(r+d+1)/2}}{x_i^{1/2}-x_i^{-1/2}}\left[\left(\frac{r+1}{2}\right)^{n}\right]^{+}_{D(n)}, \quad (6.7)$$

and

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\emptyset}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu} (-1)^{|\alpha|+\#(\alpha<\gamma)+s(\sigma)}$$

$$\times c^P_{(\beta|\gamma)}\Big((\alpha+r+d|\alpha)\cup\big(t_\sigma^{r+d}(\beta)|\gamma\big)\Big)_{C(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)x_i^{-d/2}$$

$$\times\Big(x_i^{n+s+(r+d+1)/2} - x_i^{-n-s-(r+d+1)/2}\Big)\left(\left(\frac{r}{2}\right)^n\right)_{B(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)x_i^{-d/2}$$

$$\times\Big(x_i^{n+s+(r+d+1)/2} - x_i^{-n-s-(r+d+1)/2}\Big)$$

$$\times\Big(x_i^{1/2} + x_i^{-1/2}\Big)\left(\left(\frac{r-1}{2}\right)^n\right)_{C(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)x_i^{-d/2}$$

$$\times\frac{x_i^{n+s+(r+d+1)/2} - x_i^{-n-s-(r+d+1)/2}}{x_i^{1/2} - x_i^{-1/2}}\left[\left(\frac{r+1}{2}\right)^n\right]^-_{D(n)}. \qquad (6.8)$$

THEOREM 6.3. *Let $n, r \in \mathbb{N}$ and $s \in \frac{1}{2}\mathbb{N}$, and let $P(x) = \sum_{i=0}^{d} a_i x^i$ be a polynomial of degree d with $a_0 = 1$. If $P(x)$ is symmetric, then we have*

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\emptyset}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu} (-1)^{|\alpha|+\#(\alpha<\gamma)+s(\sigma)}$$

$$\times c^P_{(\beta|\gamma)}\Big[(\alpha+r+d|\alpha)\cup\big(t_\sigma^{r+d}(\beta)|\gamma\big)+s^n\Big]^{\pm}_{D(n)}$$

$$= (-1)^{n(n-1)/2}\prod_{i=1}^{n} P(x_i)x_i^{-d/2}$$

$$\times\frac{x_i^{n+s+(r+d-1)/2} \pm x_i^{-n-s-(r+d-1)/2}}{x_i^{1/2} - x_i^{-1/2}}\left[\left(\frac{r+1}{2}\right)^n\right]^+_{D(n)}, \qquad (6.9)$$

and

$$\sum_{\nu=0}^{d} \sum_{\substack{\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\varnothing}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu} (-1)^{|\alpha|+\#\alpha+\#(\alpha<\gamma)+s(\sigma)+|\sigma|}$$
$$\times c^P_{(\beta|\gamma)}\Big[(\alpha+r+d|\alpha)\cup\big(t^{r+d}_\sigma(\beta)|\gamma\big)\Big]^{\pm}_{D(n)}$$
$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)\,x_i^{-d/2}$$
$$\times\big(x_i^{n+s+(r+d-1)/2} \mp x_i^{-n-s-(r+d-1)/2}\big)\bigg(\Big(\frac{r}{2}\Big)^n\bigg)_{B(n)}$$
$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)\,x_i^{-d/2}$$
$$\times\big(x_i^{n+s+(r+d-1)/2} \mp x_i^{-n-s-(r+d-1)/2}\big)$$
$$\times\big(x_i^{1/2}+x_i^{-1/2}\big)\bigg(\Big(\frac{r-1}{2}\Big)^n\bigg)_{C(n)}$$
$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n} P(x_i)\,x_i^{-d/2}$$
$$\times\frac{x_i^{n+s+(r+d-1)/2} \mp x^{-n-s-(r+d+1)/2}}{x_i^{1/2}-x_i^{-1/2}}\bigg[\Big(\frac{r+1}{2}\Big)^n\bigg]^{-}_{D(n)}. \quad (6.10)$$

If $P(x)$ is antisymmetric, then we have

$$\sum_{\nu=0}^{d} \sum_{\substack{\alpha,\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\varnothing}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu} (-1)^{|\alpha|+\#\alpha+\#(\alpha<\gamma)+s(\sigma)+|\sigma|}$$
$$\times c^P_{(\beta|\gamma)}\Big[(\alpha+r+d|\alpha)\cup\big(t^{r+d}_\sigma(\beta)|\gamma\big)+s^n\Big]^{\pm}_{D(n)}$$
$$= (-1)^{n(n-1)/2}\prod_{i=1}^{n} P(x_i)\,x_i^{-d/2}$$
$$\times\frac{x_i^{n+s+(r+d-1)/2} \mp x_i^{-n-s-(r+d-1)/2}}{x_i^{1/2}-x_i^{-1/2}}\bigg[\Big(\frac{r+1}{2}\Big)^n\bigg]^{+}_{D(n)}, \quad (6.11)$$

and

$$\sum_{\nu=0}^{d} \sum_{\substack{\gamma\subset[0,n)\\ \#\gamma=\nu\\ \alpha\cap\gamma=\emptyset}} \sum_{\substack{\beta\subset[0,d)\\ \#\beta=\nu}} \sum_{\sigma\in\mathscr{I}_\nu} (-1)^{|\alpha|+\#(\alpha<\gamma)+s(\sigma)}$$

$$\times c^{P}_{(\beta|\gamma)}\left[(\alpha+r+d|\alpha)\cup\left(t_{\sigma}^{r+d}(\beta)|\gamma\right)\right]^{\pm}_{D(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n}P(x_i)x_i^{-d/2}$$

$$\times\left(x_i^{n+s+(r+d-1)/2}\pm x_i^{-n-s-(r+d-1)/2}\right)\left(\left(\frac{r}{2}\right)^n\right)_{B(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n}P(x_i)x_i^{-d/2}$$

$$\times\left(x_i^{n+s+(r+d-1)/2}\pm x_i^{-n-s-(r+d-1)/2}\right)$$

$$\times\left(x_i^{1/2}+x_i^{-1/2}\right)\left(\left(\frac{r-1}{2}\right)^n\right)_{C(n)}$$

$$= (-1)^{n(n+1)/2}\prod_{i=1}^{n}P(x_i)x_i^{-d/2}$$

$$\times\frac{x_i^{n+s+(r+d-1)/2}\pm x^{-n-s-(r+d-1)/2}}{x_i^{1/2}-x_i^{-1/2}}\left[\left(\frac{r+1}{2}\right)^n\right]^{-}_{D(n)}. \quad (6.12)$$

REFERENCES

[H] R. Howe, Remarks on classical invariant theory, *Trans. Amer. Math. Soc.* **313** (1989), 539–570; erratum, *Trans. Amer. Math. Soc.* **318** (1990), 823.

[HU] R. Howe and T. Umeda, The Capelli identity, the double commutant theorem, and multiplicity-free actions, *Math. Ann.* **290** (1991), 565–619.

[I] M. Ishikawa, A remark on totally symmetric self-complementary plane partitions, preprint.

[IOW] M. Ishikawa, S. Okada, and M. Wakayama, Minor summation of Pfaffians and generalized Littlewood type formulas, *J. Algebra* **183** (1996), 193–216.

[IW1] M. Ishikawa and M. Wakayama, Minor summation of Pfaffians, *Linear and Multilinear Algebra* **39** (1995), 285–305.

[IW2] M. Ishikawa and M. Wakayama, Minor summation of Pfaffians and Schur functions identities, *Proc. Japan Acad. Ser. A* **71** (1995), 54–57.

[IW3] M. Ishikawa and M. Wakayama, Minor summation formula and applications, discrete Fourier transforms, *RIMS Kôkyû-roku* **933** (1995), 85–104.

[IW4] M. Ishikawa and M. Wakayama, Applications of minor summation formulas II, Pfaffians, and Schur polynomials, preprint.

[Kr] C. Krattenthaler, On bideterminantal formulas for characters of classical groups, preprint.

[LP] A. Lascoux and P. Pragacz, S-function series, *J. Phys. A* **21** (1988), 4105–4114.

[Ma] I. G. Macdonald, "Symmetric Functions and Hall Polynomials," 2nd ed., Oxford Mathematical Monographs, Oxford Univ. Press, London, 1995.

[NUW1] M. Noumi, T. Umeda, and M. Wakayama, A quantum dual pair $(\mathfrak{sl}_2, \mathfrak{o}_n)$ and the associated Capelli identity, *Lett. Math. Phys.* **34** (1995), 1–8.

[NUW2] N. Noumi, T. Umeda, and M. Wakayama, Dual pairs, spherical harmonics and a Capelli identity in quantum group theory, *Composition Math.* **104** (1996), 227–277.

[Ok1] S. Okada, On the generating function for certain classes of plane partitions, *J. Combin. Theory Ser. A* **51** (1989), 1–23.

[Ok2] S. Okada, Applications of minor-summation formulas to rectangular-shaped representations of classical groups, preprint.

[Pr] R. A. Proctor, Young tableaux, Gelfand patterns, and branching rules for classical groups, preprint.

[St] R. P. Stanley, "Enumerative Combinatorics," Vol. I, Wadsworth, Monterey, CA, 1986.

[Ste] J. Stembridge, Nonintersecting paths, Pfaffians and plane partitions, *Adv. Math.* **83** (1990), 96–131.

[Su] T. Sundquist, Pfaffians, involutions, and Schur functions, University of Minnesota, PhD thesis.

[Wy] H. Weyl, "The Classical Groups, Their Invariants and Representations," 2nd Ed., Princeton Univ. Press, Princeton, NJ, 1946.

[YW] M. Yang and B. G. Wybourne, New S function series and non-compact Lie groups, *J. Phys. A* **19** (1986), 3513–3525.