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Generalizations of Cauchy's determinant and Schur's Pfaffian

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Abstract

We present several identities of Cauchy-type determinants and Schur-type Pfaffians involving generalized Vandermonde determinants, which generalize Cauchy's determinant $\det(1/(x_i + y_j))$ and Schur's Pfaffian $\mathrm{Pf}((x_j - x_i)/(x_j + x_i))$. Some special cases of these identities are given by S. Okada and T. Sundquist. As an application, we give a relation for the Littlewood–Richardson coefficients involving a rectangular partition.

Keywords: Pfaffian; Cauchy's Determinant; Plücker relations; Schur functions

1. Introduction

Computations of determinants and Pfaffians are of great importance not only in many branches of mathematics but also in physics. Some people need relations among minors or sub-Pfaffians of a general matrix, others have to evaluate special determinants or Pfaf-

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fians. In enumerative combinatorics and representation theory, a central role is played by Cauchy's determinant identity [2]

$$\det\left(\frac{1}{x_i+y_j}\right)_{1\leqslant i,j\leqslant n}=\frac{\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)(y_j-y_i)}{\prod_{i,j=1}^{n}(x_i+y_j)}, \tag{1.1}$$

and Schur's Pfaffian identity [17]

$$\operatorname{Pf}\left(\frac{x_j-x_i}{x_j+x_i}\right)_{1\leqslant i,j\leqslant 2n}=\prod_{1\leqslant i<j\leqslant 2n}\frac{x_j-x_i}{x_j+x_i}. \tag{1.2}$$

The reader is referred to [1,4,6,9,13,14,18,19] for some recent variations and generalizations with their applications of (1.2) and (1.1). Besides, Krattenthaler [8] has given a comprehensive survey of determinant evaluations.

In the same vein, we shall give several identities of Cauchy-type determinants and Schur-type Pfaffians whose entries involve two kinds of generalized Vandermonde determinants. Let $\boldsymbol{x}=(x_1,\dots,x_n)$ and $\boldsymbol{a}=(a_1,\dots,a_n)$ be two vectors of variables of length n. Let p and q be two nonnegative integers such that $p+q=n$. Denote by $V^{p,q}(\boldsymbol{x};\boldsymbol{a})$ the $n\times n$ matrix with ith row

$$\left(1,x_i,\dots,x_i^{p-1},a_i,a_ix_i,\dots,a_ix_i^{q-1}\right),$$

and $W^n(\boldsymbol{x};\boldsymbol{a})$ the $n\times n$ matrix with ith row

$$\left(1+a_ix_i^{n-1},x_i+a_ix_i^{n-2},\dots,x_i^{n-1}+a_i\right).$$

For example, if $q=0$, then $V^{n,0}(\boldsymbol{x};\boldsymbol{a})=(x_i^{j-1})_{1\leqslant i,j\leqslant n}$ is the usual Vandermonde matrix and $\det V^{n,0}(\boldsymbol{x};\boldsymbol{a})=\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)$. If $p=q=1$, then $\det V^{1,1}(\boldsymbol{x};\boldsymbol{a})=a_2-a_1$, while the matrices $V^{3,2}(\boldsymbol{x};\boldsymbol{a})$ and $W^5(\boldsymbol{x};\boldsymbol{a})$ can be visualized as follows:

$$V^{3,2}(\boldsymbol{x};\boldsymbol{a})=\begin{pmatrix}1&x_1&x_1^2&a_1&a_1x_1\\1&x_2&x_2^2&a_2&a_2x_2\\1&x_3&x_3^2&a_3&a_3x_3\\1&x_4&x_4^2&a_4&a_4x_4\\1&x_5&x_5^2&a_5&a_5x_5\end{pmatrix},$$

$$W^5(\boldsymbol{x};\boldsymbol{a})=\begin{pmatrix}1+a_1x_1^4&x_1+a_1x_1^3&x_1^2+a_1x_1^2&x_1^3+a_1x_1&x_1^4+a_1\\1+a_2x_2^4&x_2+a_2x_2^3&x_2^2+a_2x_2^2&x_2^3+a_2x_2&x_2^4+a_2\\1+a_3x_3^4&x_3+a_3x_3^3&x_3^2+a_3x_3^2&x_3^3+a_3x_3&x_3^4+a_3\\1+a_4x_4^4&x_4+a_4x_4^3&x_4^2+a_4x_4^2&x_4^3+a_4x_4&x_4^4+a_4\\1+a_5x_5^4&x_5+a_5x_5^3&x_5^2+a_5x_5^2&x_5^3+a_5x_5&x_5^4+a_5\end{pmatrix}.$$

The main purpose of this paper is to prove the following identities for the determinants and Pfaffians whose entries involve these generalized Vandermonde determinants.

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Theorem 1.1. (a) *Let n be a positive integer and let p and q be nonnegative integers. For six vectors of variables*

$$\boldsymbol{x}=(x_1,\dots,x_n),\quad \boldsymbol{y}=(y_1,\dots,y_n),\quad \boldsymbol{a}=(a_1,\dots,a_n),\quad \boldsymbol{b}=(b_1,\dots,b_n),$$
$$\boldsymbol{z}=(z_1,\dots,z_{p+q}),\quad \boldsymbol{c}=(c_1,\dots,c_{p+q}),$$

we have

$$\det\left(\frac{\det V^{p+1,q+1}(x_i,y_j,\boldsymbol{z};a_i,b_j,\boldsymbol{c})}{y_j-x_i}\right)_{1\leqslant i,j\leqslant n} = \frac{(-1)^{n(n-1)/2}}{\prod_{i,j=1}^{n}(y_j-x_i)}\det V^{p,q}(\boldsymbol{z};\boldsymbol{c})^{n-1}\det V^{n+p,n+q}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z};\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}). \tag{1.3}$$

(b) *Let n be a positive integer and let p, q, r, s be nonnegative integers. For seven vectors of variables*

$$\boldsymbol{x}=(x_1,\dots,x_{2n}),\quad \boldsymbol{a}=(a_1,\dots,a_{2n}),\quad \boldsymbol{b}=(b_1,\dots,b_{2n}),$$
$$\boldsymbol{z}=(z_1,\dots,z_{p+q}),\quad \boldsymbol{c}=(c_1,\dots,c_{p+q}),$$
$$\boldsymbol{w}=(w_1,\dots,w_{r+s}),\quad \boldsymbol{d}=(d_1,\dots,d_{r+s}),$$

we have

$$\begin{aligned}&\mathrm{Pf}\left(\frac{\det V^{p+1,q+1}(x_i,x_j,\boldsymbol{z};a_i,a_j,\boldsymbol{c})\det V^{r+1,s+1}(x_i,x_j,\boldsymbol{w};b_i,b_j,\boldsymbol{d})}{x_j-x_i}\right)_{1\leqslant i,j\leqslant 2n}\\ &= \frac{1}{\prod_{1\leqslant i<j\leqslant 2n}(x_j-x_i)}\det V^{p,q}(\boldsymbol{z};\boldsymbol{c})^{n-1}\det V^{r,s}(\boldsymbol{w};\boldsymbol{d})^{n-1}\det V^{n+p,n+q}(\boldsymbol{x},\boldsymbol{z};\boldsymbol{a},\boldsymbol{c})\\ &\quad\times\det V^{n+r,n+s}(\boldsymbol{x},\boldsymbol{w};\boldsymbol{b},\boldsymbol{d}).\end{aligned} \tag{1.4}$$

(c) *Let n be a positive integer and let p be a nonnegative integer. For six vectors of variables*

$$\boldsymbol{x}=(x_1,\dots,x_n),\quad \boldsymbol{y}=(y_1,\dots,y_n),\quad \boldsymbol{a}=(a_1,\dots,a_n),\quad \boldsymbol{b}=(b_1,\dots,b_n),$$
$$\boldsymbol{z}=(z_1,\dots,z_p),\quad \boldsymbol{c}=(c_1,\dots,c_p),$$

we have

$$\det\left(\frac{\det W^{p+2}(x_i,y_j,\boldsymbol{z};a_i,b_j,\boldsymbol{c})}{(y_j-x_i)(1-x_iy_j)}\right)_{1\leqslant i,j\leqslant n} = \frac{1}{\prod_{i,j=1}^{n}(y_j-x_i)(1-x_iy_j)}\det W^{p}(\boldsymbol{z};\boldsymbol{c})^{n-1}\det W^{2n+p}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z};\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}). \tag{1.5}$$

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(d) *Let n be a positive integer and let p and q be nonnegative integers. For seven vectors of variables*

$$\begin{gathered}
\boldsymbol{x}=(x_1,\dots,x_{2n}),\quad \boldsymbol{a}=(a_1,\dots,a_{2n}),\quad \boldsymbol{b}=(b_1,\dots,b_{2n}),\\
\boldsymbol{z}=(z_1,\dots,z_p),\quad \boldsymbol{c}=(c_1,\dots,c_p),\\
\boldsymbol{w}=(w_1,\dots,w_q),\quad \boldsymbol{d}=(d_1,\dots,d_q),
\end{gathered}$$

we have

$$\begin{aligned}
&\operatorname{Pf}\left(\frac{\det W^{p+2}(x_i,x_j,\boldsymbol{z};a_i,a_j,\boldsymbol{c})\det W^{q+2}(x_i,x_j,\boldsymbol{w};b_i,b_j,\boldsymbol{d})}{(x_j-x_i)(1-x_ix_j)}\right)_{1\leqslant i,j\leqslant 2n}\\
&\quad=\frac{1}{\prod_{1\leqslant i<j\leqslant 2n}(x_j-x_i)(1-x_ix_j)}\det W^p(\boldsymbol{z};\boldsymbol{c})^{n-1}\det W^q(\boldsymbol{w};\boldsymbol{d})^{n-1}\\
&\qquad\times\det W^{2n+p}(\boldsymbol{x},\boldsymbol{z};\boldsymbol{a},\boldsymbol{c})\det W^{2n+q}(\boldsymbol{x},\boldsymbol{w};\boldsymbol{b},\boldsymbol{d}).
\end{aligned}\tag{1.6}$$

These identities were conjectured by one of the authors [15]. If we put $p=q=0$ in (1.3) or $p=q=r=s=0$ in (1.4), then the identities read

$$\det\left(\frac{b_j-a_i}{y_j-x_i}\right)_{1\leqslant i,j\leqslant n}=\frac{(-1)^{n(n-1)/2}}{\prod_{i,j=1}^n(y_j-x_i)}\det V^{n,n}(\boldsymbol{x},\boldsymbol{y};\boldsymbol{a},\boldsymbol{b}),\tag{1.7}$$

$$\operatorname{Pf}\left(\frac{(a_j-a_i)(b_j-b_i)}{x_j-x_i}\right)_{1\leqslant i,j\leqslant 2n}=\frac{1}{\prod_{1\leqslant i<j\leqslant 2n}(x_j-x_i)}\det V^{n,n}(\boldsymbol{x};\boldsymbol{a})\det V^{n,n}(\boldsymbol{x};\boldsymbol{b}).\tag{1.8}$$

These particular cases, as well as the identities (1.5) with $p=0$ and (1.6) with $p=q=0$, are first given by S. Okada [13, Theorems 4.2, 4.7, 4.3, 4.4] in his study of rectangular-shaped representations of classical groups. Another special case of the identity (1.5) with $p=1$ is given in [14] and applied to the enumeration of vertically and horizontally symmetric alternating sign matrices. These special cases are the starting point of our study.

Under the specialization

$$\begin{gathered}
x_i\leftarrow x_i^2,\qquad y_i\leftarrow y_i^2,\qquad z_i\leftarrow z_i^2,\qquad w_i\leftarrow w_i^2,\\
a_i\leftarrow x_i,\qquad b_i\leftarrow y_i,\qquad c_i\leftarrow z_i,\qquad d_i\leftarrow w_i,
\end{gathered}$$

one can deduce from (1.3) and (1.4) the following identities:

$$\begin{aligned}
&\det\left(\frac{s_{\delta(k)}(x_i,y_j,\boldsymbol{z})}{x_i+y_j}\right)_{1\leqslant i,j\leqslant n}\\
&\quad=\frac{\prod_{1\leqslant i<j\leqslant n}(x_j-x_i)(y_j-y_i)}{\prod_{i,j=1}^n(x_i+y_j)}s_{\delta(k)}(\boldsymbol{z})^{n-1}s_{\delta(k)}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}),
\end{aligned}\tag{1.9}$$

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$$\operatorname{Pf}\left(\frac{x_j-x_i}{x_j+x_i}s_{\delta(k)}(x_i,x_j,\boldsymbol{z})s_{\delta(l)}(x_i,x_j,\boldsymbol{w})\right)_{1\leqslant i,j\leqslant 2n}$$
$$=\prod_{1\leqslant i<j\leqslant 2n}\frac{x_j-x_i}{x_j+x_i}s_{\delta(k)}(\boldsymbol{z})^{n-1}s_{\delta(l)}(\boldsymbol{w})^{n-1}s_{\delta(k)}(\boldsymbol{x},\boldsymbol{z})s_{\delta(l)}(\boldsymbol{x},\boldsymbol{w}),\tag{1.10}$$

where s_λ denotes the Schur function corresponding to a partition λ and $\delta(k)=(k,k-1,\dots,1)$ denotes the staircase partition. If we take $k=0$ in (1.9) and $k=l=0$ in (1.10), we obtain Cauchy's determinant identity (1.1) and Schur's Pfaffian identity (1.2). Another special case of (1.9) with $k=1$ is the rational case of Frobenius' identity [3]. Also, if we take $k=l=1$ in (1.10), we obtain the rational case of an elliptic generalization of (1.2) given in [16].

This paper is organized as follows. Sections 2 and 3 are devoted to the proof of Theorem 1.1. In Section 2, we prove the identity (1.4) by using the Pfaffian version of Desnanot–Jacobi formula and induction. In Section 3, we give a homogeneous version of the identity (1.4) and derive the other three identities (1.3), (1.5) and (1.6). A variation of the main identities is given in Section 4, and another instance of a Cauchy-type determinant identity is presented in Section 5. Also we present a formula expressing the determinant of $V^{n,n}$ in terms of the hyperpfaffian. In the last section, we give an application of the identity (1.4) to the Littlewood–Richardson coefficients.

Here we recall the definition of Pfaffians. Given a $2n\times 2n$ skew-symmetric matrix $A=(a_{ij})_{1\leqslant i,j\leqslant 2n}$, the Pfaffian of A is defined by

$$\operatorname{Pf}(A)=\sum_{\sigma}\operatorname{sgn}(\sigma)a_{\sigma(1),\sigma(2)}a_{\sigma(3),\sigma(4)}\cdots a_{\sigma(2n-1),\sigma(2n)},$$

where σ runs over all permutations of $2n$ letters $1,2,\dots,2n$ satisfying

$$\sigma(1)<\sigma(2),\qquad \sigma(3)<\sigma(4),\qquad \dots,\qquad \sigma(2n-1)<\sigma(2n),$$
$$\sigma(1)<\sigma(3)<\cdots<\sigma(2n-1).$$

2. Proof of the identity (1.4) in Theorem 1.1

In this section, we give a proof of the identity (1.4). First we show (1.4) in the special case where $n=2$ by using induction. Then we apply the Desnanot–Jacobi formula for Pfaffians to reduce the proof of the general case to this special case.

First we prove the case of $n=2$ by induction on $p+q+r+s$.

Proposition 2.1. *Let p, q, r and s be nonnegative integers. For vectors of variables $\boldsymbol{x}$, $\boldsymbol{a}$, $\boldsymbol{b}$ of length 4, $\boldsymbol{z}$, $\boldsymbol{c}$ of length $p+q$, and $\boldsymbol{w}$, $\boldsymbol{d}$ of length $r+s$, we have*

$$\operatorname{Pf}\left(\frac{\det V^{p+1,q+1}(x_i,x_j,\boldsymbol{z};a_i,a_j,\boldsymbol{c})\det V^{r+1,s+1}(x_i,x_j,\boldsymbol{w};b_i,b_j,\boldsymbol{d})}{x_j-x_i}\right)_{1\leqslant i,j\leqslant 4}$$

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$$= \frac{1}{\prod_{1\leqslant i<j\leqslant 4}(x_j - x_i)} \det V^{p,q}(\boldsymbol{z};\boldsymbol{c}) \det V^{r,s}(\boldsymbol{w};\boldsymbol{d}) \\ \times \det V^{p+2,q+2}(\boldsymbol{x},\boldsymbol{z};\boldsymbol{a},\boldsymbol{c}) \det V^{r+2,s+2}(\boldsymbol{x},\boldsymbol{w};\boldsymbol{b},\boldsymbol{d}). \tag{2.1}$$

In the induction step of the proof, we need relations between $\det V^{p,q}$ and $\det V^{p-1,q}$ (or $\det V^{q,p}$).

Lemma 2.2. (1) *If $p \geqslant q$ and $p \geqslant 1$, then we have*

$$\det V^{p,q}(\boldsymbol{x};\boldsymbol{a}) = \prod_{i=1}^{p+q-1}(x_{p+q} - x_i)\cdot \det V^{p-1,q}(x_1,\dots,x_{p+q-1};a'_1,\dots,a'_{p+q-1}), \tag{2.2}$$

where we put

$$a'_i = \frac{a_i - a_{p+q}}{x_i - x_{p+q}} \quad (1\leqslant i\leqslant p+q-1).$$

(2) *For nonnegative integers p and q, we have*

$$\det V^{p,q}(\boldsymbol{x};\boldsymbol{a}) = (-1)^{pq}\prod_{i=1}^{p+q} a_i \cdot \det V^{q,p}\big(\boldsymbol{x};\boldsymbol{a}^{-1}\big), \tag{2.3}$$

where $\boldsymbol{a}^{-1} = (a_1^{-1},\dots,a_{p+q}^{-1})$.

Proof. (1) We put $m = p+q$. By subtracting the ith column multiplied by a_m from the $(p+i)$th column for $i = 1,\dots,q$, and by subtracting the ith column multiplied by x_m from the $(i+1)$th column for $i = p-1,\dots,1$, we obtain

$$\det V^{p,q}(\boldsymbol{x};\boldsymbol{a}) \\ = \det\begin{pmatrix} 1 & x_1 - x_m & \cdots & (x_1-x_m)x_1^{p-2} & a_1 - a_m & \cdots & (a_1-a_m)x_1^{q-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & x_{m-1}-x_m & \cdots & (x_{m-1}-x_m)x_{m-1}^{p-2} & a_{m-1}-a_m & \cdots & (a_{m-1}-a_m)x_{m-1}^{q-1} \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \\ = (-1)^{m+1}\prod_{k=1}^{m-1}(x_k - x_m) \\ \times \det\begin{pmatrix} 1 & x_1 & \cdots & x_1^{p-2} & a'_1 & a'_1x_1 & \cdots & a'_1x_1^{q-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{m-1} & \cdots & x_{m-1}^{p-2} & a'_{m-1} & a'_{m-1}x_{m-1} & \cdots & a'_{m-1}x_{m-1}^{q-1} \end{pmatrix} \\ = \prod_{k=1}^{p+q-1}(x_{p+q} - x_k)\cdot \det V^{p-1,q}\big(x_1,\dots,x_{p+q-1};a'_1,\dots,a'_{p+q-1}\big).$$

(2) As before we put $m = p + q$. By performing an appropriate permutation of the columns and by dividing the ith row by a_i, we obtain

$$\begin{aligned}
&\det V^{p,q}(\boldsymbol{x};\boldsymbol{a}) \\
&\quad = (-1)^{pq} \det \begin{pmatrix} a_1 & a_1x_1 & \cdots & a_1x_1^{q-1} & 1 & x_1 & \cdots & x_1^{p-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_m & a_mx_m & \cdots & a_mx_m^{q-1} & 1 & x_m & \cdots & x_m^{p-1} \end{pmatrix} \\
&\quad = (-1)^{pq} \prod_{k=1}^{p+q} a_k \cdot \det \begin{pmatrix} 1 & x_1 & \cdots & x_1^{q-1} & a_1^{-1} & a_1^{-1}x_1 & \cdots & a_1^{-1}x_1^{p-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_m & \cdots & x_m^{q-1} & a_m^{-1} & a_m^{-1}x_m & \cdots & a_m^{-1}x_m^{q-1} \end{pmatrix} \\
&\quad = (-1)^{pq} \prod_{k=1}^{p+q} a_k \cdot \det V^{q,p}\bigl(\boldsymbol{x};\boldsymbol{a}^{-1}\bigr).
\end{aligned}$$

This completes the proof of the lemma. □

Proof of Proposition 2.1. We prove (2.1) by induction on $p+q+r+s$. If $p+q+r+s=0$, i.e., $p=q=r=s=0$, then one can easily check the equality in (2.1) by a direct computation.

Suppose $p+q+r+s>0$. By symmetry, we may assume $p+q>0$ without loss of generality. First we consider the case where $p \geqslant q$. Using the relation (2.2) in Lemma 2.2, we have

$$\begin{aligned}
&\det V^{p+1,q+1}(x_i, x_j, \boldsymbol{z}; a_i, a_j, \boldsymbol{c}) \\
&\quad = (z_{p+q}-x_i)(z_{p+q}-x_j) \prod_{k=1}^{p+q-1} (z_{p+q}-z_k) \cdot \det V^{p,q+1}\bigl(x_i, x_j, \tilde{\boldsymbol{z}}; a_i', a_j', \boldsymbol{c}'\bigr),
\end{aligned}$$

where $\tilde{\boldsymbol{z}} = (z_1, \dots, z_{p+q-1})$ and a_i', a_j' and $\boldsymbol{c}' = (c_1', \dots, c_{p+q-1}')$ are given by

$$a_k' = \frac{a_k - c_{p+q}}{x_k - z_{p+q}} \quad (k=i,j), \qquad c_l' = \frac{c_l - c_{p+q}}{z_l - z_{p+q}} \quad (1 \leqslant l \leqslant p+q-1).$$

Hence we have

$$\begin{aligned}
&\mathrm{Pf}\left(\frac{\det V^{p+1,q+1}(x_i, x_j, \boldsymbol{z}; a_i, a_j, \boldsymbol{c}) \det V^{r+1,s+1}(x_i, x_j, \boldsymbol{w}; b_i, b_j, \boldsymbol{d})}{x_j - x_i}\right)_{1\leqslant i,j\leqslant 4} \\
&= \prod_{i=1}^{4}(z_{p+q}-x_i) \prod_{i=1}^{p+q-1} (z_{p+q}-z_i)^2 \\
&\quad \times \mathrm{Pf}\left(\frac{\det V^{p,q+1}(x_i, x_j, \tilde{\boldsymbol{z}}; a_i', a_j', \boldsymbol{c}') \det V^{r+1,s+1}(x_i, x_j, \boldsymbol{w}; b_i, b_j, \boldsymbol{d})}{x_j - x_i}\right)_{1\leqslant i,j\leqslant 4}
\end{aligned}$$

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by the induction hypothesis,

$$
\begin{aligned}
&= \prod_{i=1}^{4}(z_{p+q} - x_i) \prod_{i=1}^{p+q-1}(z_{p+q} - z_i)^2 \\
&\quad \times \frac{1}{\prod_{1\leqslant i<j\leqslant 4}(x_j - x_i)} \det V^{p-1,q}\big(\tilde{z}; \boldsymbol{c}'\big) \det V^{r,s}(\boldsymbol{w}; \boldsymbol{d}) \\
&\quad \times \det V^{p+1,q+2}\big(x_1, x_2, x_3, x_4, \tilde{z}; a_1', a_2', a_3', a_4', \boldsymbol{c}'\big) \det V^{r+2,s+2}(\boldsymbol{x}, \boldsymbol{w}; \boldsymbol{b}, \boldsymbol{d})
\end{aligned}
$$

by using the relation (2.2) again,

$$
\begin{aligned}
&= \frac{1}{\prod_{1\leqslant i<j\leqslant 4}(x_j - x_i)} \det V^{p,q}(\boldsymbol{z}; \boldsymbol{c}) \det V^{r,s}(\boldsymbol{w}; \boldsymbol{d}) \det V^{p+2,q+2}(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{a}, \boldsymbol{c}) \\
&\quad \times \det V^{r+2,s+2}(\boldsymbol{x}, \boldsymbol{w}; \boldsymbol{b}, \boldsymbol{d}).
\end{aligned}
$$

If $p < q$, then we use the relation (2.3) in Lemma 2.2 and the case we have just proven. Then we see that

$$
\begin{aligned}
&\mathrm{Pf}\left(\frac{\det V^{p+1,q+1}(x_i, x_j, \boldsymbol{z}; a_i, a_j, \boldsymbol{c}) \det V^{r+1,s+1}(x_i, x_j, \boldsymbol{w}; b_i, b_j, \boldsymbol{d})}{x_j - x_i}\right)_{1\leqslant i,j\leqslant 4} \\
&\quad = \mathrm{Pf}\left(\frac{a_i a_j \prod_{k=1}^{p+q} c_k \cdot \det V^{q+1,p+1}(x_i, x_j, z; a_i^{-1}, a_j^{-1}, \boldsymbol{c}^{-1}) \det V^{r+1,s+1}(x_i, x_j, \boldsymbol{w}; b_i, b_j, \boldsymbol{d})}{x_j - x_i}\right)_{1\leqslant i,j\leqslant 4} \\
&\quad = \prod_{i=1}^{4} a_i \prod_{k=1}^{p+q} c_k^2 \frac{1}{\prod_{1\leqslant i<j\leqslant 2n}(x_j - x_i)} \det V^{q,p}\big(\boldsymbol{z}; \boldsymbol{c}^{-1}\big) \det V^{r,s}(\boldsymbol{w}; \boldsymbol{d}) \\
&\qquad \times \det V^{q+2,p+2}\big(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{a}^{-1}, \boldsymbol{c}^{-1}\big) \det V^{r+2,s+2}(\boldsymbol{x}, \boldsymbol{w}; \boldsymbol{b}, \boldsymbol{d}) \\
&\quad = \frac{1}{\prod_{1\leqslant i<j\leqslant 4}(x_j - x_i)} \det V^{p,q}(\boldsymbol{z}; \boldsymbol{c}) \det V^{r,s}(\boldsymbol{w}; \boldsymbol{d}) \det V^{p+2,q+2}(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{a}, \boldsymbol{c}) \\
&\qquad \times \det V^{r+2,s+2}(\boldsymbol{x}, \boldsymbol{w}; \boldsymbol{b}, \boldsymbol{d}).
\end{aligned}
$$

This completes the proof of Proposition 2.1. □

Here we recall the Desnanot–Jacobi formula for determinants and Pfaffians. Given a square matrix A and indices $i_1, \ldots, i_r, j_1, \ldots, j_r$, we denote by $A^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}$ the matrix obtained by removing the rows $i_1, \ldots, i_r$ and the columns $j_1, \ldots, j_r$ of A.

Lemma 2.3.

(1) *If A is a square matrix, then we have*

$$
\det A_1^1 \cdot \det A_2^2 - \det A_2^1 \cdot \det A_1^2 = \det A \cdot \det A_{1,2}^{1,2}. \tag{2.4}
$$

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(2) *If A is a skew-symmetric matrix A, then we have*

$$\operatorname{Pf} A_{1,2}^{1,2} \cdot \operatorname{Pf} A_{3,4}^{3,4} - \operatorname{Pf} A_{1,3}^{1,3} \cdot \operatorname{Pf} A_{2,4}^{2,4} + \operatorname{Pf} A_{1,4}^{1,4} \cdot \operatorname{Pf} A_{2,3}^{2,3} = \operatorname{Pf} A \cdot \operatorname{Pf} A_{1,2,3,4}^{1,2,3,4}. \tag{2.5}$$

This Pfaffian analogue of the Desnanot–Jacobi formula is given in [7] and [6].

If $\boldsymbol{x} = (x_1, \dots, x_n)$ is a vector of variables and $1 \leqslant i_1 < \dots < i_r \leqslant n$ are indices, we denote by $\boldsymbol{x}^{(i_1,\dots,i_r)}$ the vector obtained from $\boldsymbol{x}$ by removing the variables $x_{i_1}, \dots, x_{i_r}$.

Proof of the identity in (1.4) in Theorem 1.1. We proceed by induction on n. If $n = 1$, then there is nothing to prove, and, if $n = 2$, we already proved (1.4) in Proposition 2.1.

Suppose $n \geqslant 3$. Apply the Desnanot–Jacobi formula for Pfaffians (2.5) in Lemma 2.3 to the skew-symmetric matrix

$$A = \left(\frac{\det V^{p+1,q+1}(x_i, x_j, \boldsymbol{z}; a_i, a_j, \boldsymbol{c}) \det V^{r+1,s+1}(x_i, x_j, \boldsymbol{w}; b_i, b_j, \boldsymbol{d})}{x_j - x_i} \right)_{1 \leqslant i,j \leqslant 2n}.$$

Then the induction hypothesis tells us that, for $1 \leqslant k < l \leqslant 4$, we have

$$\begin{aligned} \operatorname{Pf} A_{k,l}^{k,l} = {} & \frac{1}{(x_{l'} - x_{k'}) \prod_{i=5}^{2n} (x_i - x_{k'})(x_i - x_{l'}) \prod_{5 \leqslant i < j \leqslant 2n} (x_j - x_i)} \\ & \times \det V^{p,q}(\boldsymbol{z}; \boldsymbol{c})^{n-2} \det V^{r,s}(\boldsymbol{w}; \boldsymbol{d})^{n-2} \det V^{n+p-1,n+q-1}\big(\boldsymbol{x}^{(k,l)}, \boldsymbol{z}; \boldsymbol{a}^{(k,l)}, \boldsymbol{c}\big) \\ & \times \det V^{n+r-1,n+s-1}\big(\boldsymbol{x}^{(k,l)}, \boldsymbol{w}; \boldsymbol{b}^{(k,l)}, \boldsymbol{d}\big), \end{aligned}$$

where k' and l' are the indices satisfying $\{k, l, k', l'\} = \{1, 2, 3, 4\}$ and $k < l$, $k' < l'$, and

$$\begin{aligned} \operatorname{Pf} A_{1,2,3,4}^{1,2,3,4} = {} & \frac{1}{\prod_{5 \leqslant i < j \leqslant 2n} (x_j - x_i)} \det V^{p,q}(\boldsymbol{z}; \boldsymbol{c})^{n-3} \det V^{r,s}(\boldsymbol{w}; \boldsymbol{d})^{n-3} \\ & \times \det V^{n+p-2,n+q-2}\big(\boldsymbol{x}^{(1,2,3,4)}, \boldsymbol{z}; \boldsymbol{a}^{(1,2,3,4)}, \boldsymbol{c}\big) \\ & \times \det V^{n+r-2,n+s-2}\big(\boldsymbol{x}^{(1,2,3,4)}, \boldsymbol{w}; \boldsymbol{b}^{(1,2,3,4)}, \boldsymbol{d}\big). \end{aligned}$$

Hence, by applying (2.5) and canceling the common factors, we see that, in order to prove (1.4), it suffices to show

$$\begin{aligned} & \frac{\det V^{n+p-1,n+q-1}(\boldsymbol{x}^{(1,2)}, \boldsymbol{z}; \boldsymbol{a}^{(1,2)}, \boldsymbol{c}) \det V^{n+r-1,n+s-1}(\boldsymbol{x}^{(1,2)}, \boldsymbol{w}; \boldsymbol{b}^{(1,2)}, \boldsymbol{d})}{x_2 - x_1} \\ & \times \frac{\det V^{n+p-1,n+q-1}(\boldsymbol{x}^{(3,4)}, \boldsymbol{z}; \boldsymbol{a}^{(3,4)}, \boldsymbol{c}) \det V^{n+r-1,n+s-1}(\boldsymbol{x}^{(3,4)}, \boldsymbol{w}; \boldsymbol{b}^{(3,4)}, \boldsymbol{d})}{x_4 - x_3} \\ & - \frac{\det V^{n+p-1,n+q-1}(\boldsymbol{x}^{(1,3)}, \boldsymbol{z}; \boldsymbol{a}^{(1,3)}, \boldsymbol{c}) \det V^{n+r-1,n+s-1}(\boldsymbol{x}^{(1,3)}, \boldsymbol{w}; \boldsymbol{b}^{(1,3)}, \boldsymbol{d})}{x_3 - x_1} \\ & \times \frac{\det V^{n+p-1,n+q-1}(\boldsymbol{x}^{(2,4)}, \boldsymbol{z}; \boldsymbol{a}^{(2,4)}, \boldsymbol{c}) \det V^{n+r-1,n+s-1}(\boldsymbol{x}^{(2,4)}, \boldsymbol{w}; \boldsymbol{b}^{(2,4)}, \boldsymbol{d})}{x_4 - x_2} \end{aligned}$$

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$$
\begin{aligned}
&+ \frac{\det V^{n+p-1,n+q-1}(\boldsymbol{x}^{(1,4)}, \boldsymbol{z}; \boldsymbol{a}^{(1,4)}, \boldsymbol{c}) \det V^{n+r-1,n+s-1}(\boldsymbol{x}^{(1,4)}, \boldsymbol{w}; \boldsymbol{b}^{(1,4)}, \boldsymbol{d})}{x_4 - x_1} \\
&\times \frac{\det V^{n+p-1,n+q-1}(\boldsymbol{x}^{(2,3)}, \boldsymbol{z}; \boldsymbol{a}^{(2,3)}, \boldsymbol{c}) \det V^{n+r-1,n+s-1}(\boldsymbol{x}^{(2,3)}, \boldsymbol{w}; \boldsymbol{b}^{(2,3)}, \boldsymbol{d})}{x_3 - x_2} \\
= {}& \det V^{n+p-2,n+q-2}\bigl(\boldsymbol{x}^{(1,2,3,4)}, \boldsymbol{z}; \boldsymbol{a}^{(1,2,3,4)}, \boldsymbol{c}\bigr) \\
&\times \det V^{n+r-2,n+s-2}\bigl(\boldsymbol{x}^{(1,2,3,4)}, \boldsymbol{w}; \boldsymbol{b}^{(1,2,3,4)}, \boldsymbol{d}\bigr) \\
&\times \frac{\det V^{n+p,n+q}(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{a}, \boldsymbol{c}) \det V^{n+r,n+s}(\boldsymbol{x}, \boldsymbol{w}; \boldsymbol{b}, \boldsymbol{d})}{\prod_{1 \leqslant i<j \leqslant 4}(x_j - x_i)}.
\end{aligned}
$$

This is equivalent to the identity (2.1) with $\boldsymbol{z}, \boldsymbol{c}, \boldsymbol{w}, \boldsymbol{d}$ replaced by

$$
\boldsymbol{z} \leftarrow \bigl(\boldsymbol{x}^{(1,2,3,4)}, \boldsymbol{z}\bigr), \quad \boldsymbol{c} \leftarrow \bigl(\boldsymbol{a}^{(1,2,3,4)}, \boldsymbol{c}\bigr), \quad \boldsymbol{w} \leftarrow \bigl(\boldsymbol{x}^{(1,2,3,4)}, \boldsymbol{w}\bigr), \quad \boldsymbol{d} \leftarrow \bigl(\boldsymbol{b}^{(1,2,3,4)}, \boldsymbol{d}\bigr),
$$

respectively, and it is proven in Proposition 2.1. This completes the proof of (1.4). □

Remark 2.4. We can also reduce the proof of the other identities (1.3), (1.5) and (1.6) in Theorem 1.1 to the case of $n = 2$ with the help of the Desnanot–Jacobi formulae. It is easy to show the case of $n = 2$ of (1.3) by using the relations in Lemma 2.2 and induction on $p + q$. Also we can prove (1.5) (respectively (1.6)) in the case of $n = 2$, by regarding both sides as polynomials in z_p and showing that the values coincide at $(2p + 3)$ distinct points $z_1, \dots, z_{p-1}, z_1^{-1}, \dots, z_{p-1}^{-1}, x_1, x_2, y_1, y_2$ and -1 (respectively $z_1, \dots, z_{p-1}, z_1^{-1}, \dots, z_{p-1}^{-1}, x_1, x_2, x_3, x_4$ and -1) with the help of induction.

But, in this paper, we adopt another method, namely, we “homogenize” the identity (1.4) and derive the other identities from this homogeneous version (3.3).

3. Proof of the identities (1.3), (1.5) and (1.6)

In this section, we give a homogeneous version of the identity (1.4), which is shown in the previous section, and derive the identities (1.3), (1.5) and (1.6) from this homogeneous version.

Throughout the remaining of this paper, we use the following notation for vectors $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$:

$$
\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \dots, x_n + y_n), \qquad \boldsymbol{x}\boldsymbol{y} = (x_1 y_1, \dots, x_n y_n),
$$

and, for integers k and l,

$$
\boldsymbol{x}^k = \bigl(x_1^k, \dots, x_n^k\bigr), \qquad \boldsymbol{x}^k \boldsymbol{y}^l = \bigl(x_1^k y_1^l, \dots, x_n^k y_n^l\bigr).
$$

We introduce a homogeneous version of the matrix $V^{p,q}(\boldsymbol{x};\boldsymbol{a})$ as follows. For vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{a}$, $\boldsymbol{b}$ of length n and nonnegative integers p, q with $p+q=n$, we define a matrix $U^{p,q}\left(\begin{smallmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{smallmatrix}\middle|\begin{smallmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{smallmatrix}\right)$ to be the $n\times n$ matrix with ith row

$$\left(a_ix_i^{p-1}, a_ix_i^{p-2}y_i, \dots, a_iy_i^{p-1}, b_ix_i^{q-1}, b_ix_i^{q-2}y_i, \dots, b_iy_i^{q-1}\right).$$

Then the following relations between $\det V^{p,q}$ and $\det U^{p,q}$ are easily shown by elementary transformations, so we omit their proofs.

Lemma 3.1.

$$\det U^{p,q}\left(\begin{matrix}\boldsymbol{x}\\ \boldsymbol{y}\end{matrix}\middle|\begin{matrix}\boldsymbol{a}\\ \boldsymbol{b}\end{matrix}\right) = \prod_{k=1}^{p+q} a_kx_k^{p-1}\cdot\det V^{p,q}\left(\boldsymbol{x}^{-1}\boldsymbol{y};\boldsymbol{a}^{-1}\boldsymbol{b}\boldsymbol{x}^{q-p}\right). \tag{3.1}$$

In particular,

$$\det V^{p,q}(\boldsymbol{x};\boldsymbol{a}) = \det U^{p,q}\left(\begin{matrix}\mathbf{1}\\ \boldsymbol{x}\end{matrix}\middle|\begin{matrix}\mathbf{1}\\ \boldsymbol{a}\end{matrix}\right), \tag{3.2}$$

where $\mathbf{1}=(1,\dots,1)$.

Now we give a homogeneous version of the identity (1.4).

Theorem 3.2. *Let n be a positive integer and let p, q, r and s be nonnegative integers. Suppose that the vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ have length $2n$, the vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ have length $p+q$, and the vectors $\boldsymbol{\zeta}$, $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ have length $r+s$. Then we have*

$$\begin{aligned}
&\operatorname{Pf}\left(\frac{\det U^{p+1,q+1}\left(\begin{matrix}x_i,x_j,\boldsymbol{\xi}\\ y_i,y_j,\boldsymbol{\eta}\end{matrix}\middle|\begin{matrix}a_i,a_j,\boldsymbol{\alpha}\\ b_i,b_j,\boldsymbol{\beta}\end{matrix}\right)\det U^{r+1,s+1}\left(\begin{matrix}x_i,x_j,\boldsymbol{\zeta}\\ y_i,y_j,\boldsymbol{\omega}\end{matrix}\middle|\begin{matrix}c_i,c_j,\boldsymbol{\gamma}\\ d_i,d_j,\boldsymbol{\delta}\end{matrix}\right)}{\det\begin{pmatrix}x_i & x_j\\ y_i & y_j\end{pmatrix}}\right)_{1\leqslant i<j\leqslant 2n}\\
&\quad=\frac{1}{\prod_{1\leqslant i<j\leqslant 2n}\det\begin{pmatrix}x_i & x_j\\ y_i & y_j\end{pmatrix}}\det U^{p,q}\left(\begin{matrix}\boldsymbol{\xi}\\ \boldsymbol{\eta}\end{matrix}\middle|\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix}\right)^{n-1}\det U^{r,s}\left(\begin{matrix}\boldsymbol{\zeta}\\ \boldsymbol{\omega}\end{matrix}\middle|\begin{matrix}\boldsymbol{\gamma}\\ \boldsymbol{\delta}\end{matrix}\right)^{n-1}\\
&\qquad\times\det U^{n+p,n+q}\left(\begin{matrix}\boldsymbol{x},\boldsymbol{\xi}\\ \boldsymbol{y},\boldsymbol{\eta}\end{matrix}\middle|\begin{matrix}\boldsymbol{a},\boldsymbol{\alpha}\\ \boldsymbol{b},\boldsymbol{\beta}\end{matrix}\right)\det U^{n+r,n+s}\left(\begin{matrix}\boldsymbol{x},\boldsymbol{\zeta}\\ \boldsymbol{y},\boldsymbol{\omega}\end{matrix}\middle|\begin{matrix}\boldsymbol{c},\boldsymbol{\gamma}\\ \boldsymbol{d},\boldsymbol{\delta}\end{matrix}\right).
\end{aligned}\tag{3.3}$$

The special case of $p=q=r=s=0$ of this identity (3.3) is given by M. Ishikawa [4, Theorem 3.1], and is one of the key ingredients of his proof of Stanley's conjecture.

The identity (1.4) is equivalent to (3.3). It follows from the relation (3.2) that (3.3) specialize to (1.4) by setting

$$\boldsymbol{x}=\boldsymbol{a}=\boldsymbol{c}=\mathbf{1}_{2n},\quad \boldsymbol{\xi}=\boldsymbol{\alpha}=\mathbf{1}_{p+q},\quad \boldsymbol{\zeta}=\boldsymbol{\gamma}=\mathbf{1}_{r+s}$$

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and renaming the variables. Here $\mathbf{1}_n$ stands for the all-one vector of length n. As we see in the following proof, we derive (3.3) from (1.4).

Proof of Theorem 3.2. In the identity (1.4), we substitute as follows:

$$\begin{gathered}\boldsymbol{x} \leftarrow \boldsymbol{x}^{-1}\boldsymbol{y}, \quad \boldsymbol{z} \leftarrow \boldsymbol{\xi}^{-1}\boldsymbol{\eta}, \quad \boldsymbol{w} \leftarrow \boldsymbol{\zeta}^{-1}\boldsymbol{\omega}, \\ \boldsymbol{a} \leftarrow \boldsymbol{a}^{-1}\boldsymbol{b}\boldsymbol{x}^{q-p}, \quad \boldsymbol{c} \leftarrow \boldsymbol{\alpha}^{-1}\boldsymbol{\beta}\boldsymbol{\xi}^{q-p}, \quad \boldsymbol{b} \leftarrow \boldsymbol{c}^{-1}\boldsymbol{d}\boldsymbol{x}^{s-r}, \quad \boldsymbol{d} \leftarrow \boldsymbol{\gamma}^{-1}\boldsymbol{\delta}\boldsymbol{\zeta}^{s-r}.\end{gathered} \tag{3.4}$$

By using the relations (3.1) and

$$\frac{y_j}{x_j} - \frac{y_i}{x_i} = x_i^{-1}x_j^{-1}\det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix},$$

we see that the (i, j) entry of the left-hand side of (1.4) under the substitution (3.4) becomes

$$\frac{(a_i a_j x_i^p x_j^p \prod_{k=1}^{p+q}\alpha_k\xi_k^p)^{-1}(c_i c_j x_i^r x_j^r \prod_{k=1}^{r+s}\gamma_k\zeta_k^r)^{-1}}{x_i^{-1}x_j^{-1}} \\ \times \frac{\det U^{p+1,q+1}\left(\begin{matrix} x_i, x_j, \boldsymbol{\xi} \\ y_i, y_j, \boldsymbol{\eta}\end{matrix}\,\middle|\,\begin{matrix} a_i, a_j, \boldsymbol{\alpha} \\ b_i, b_j, \boldsymbol{\beta}\end{matrix}\right)\det U^{r+1,s+1}\left(\begin{matrix} x_i, x_j, \boldsymbol{\zeta} \\ y_i, y_j, \boldsymbol{\omega}\end{matrix}\,\middle|\,\begin{matrix} c_i, c_j, \boldsymbol{\gamma} \\ d_i, d_j, \boldsymbol{\delta}\end{matrix}\right)}{\det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}}.$$

Hence, by noting the linearity of Pfaffian

$$\operatorname{Pf}(\lambda\alpha_i\alpha_j a_{ij})_{1\leqslant i<j\leqslant 2n} = \lambda^n \prod_{i=1}^{2n}\alpha_i \cdot \operatorname{Pf}(a_{ij})_{1\leqslant i<j\leqslant 2n},$$

we can see the Pfaffian on the left-hand side of (1.4) becomes

$$\left(\prod_{i=1}^{2n} a_i c_i x_i^{p+r-1}\right)^{-1}\left(\prod_{i=1}^{p+q}\alpha_i\xi_i^p\prod_{i=1}^{r+s}\gamma_i\zeta_i^r\right)^{-n} \\ \times \operatorname{Pf}\left(\frac{\det U^{p+1,q+1}\left(\begin{matrix} x_i, x_j, \boldsymbol{\xi} \\ y_i, y_j, \boldsymbol{\eta}\end{matrix}\,\middle|\,\begin{matrix} a_i, a_j, \boldsymbol{\alpha} \\ b_i, b_j, \boldsymbol{\beta}\end{matrix}\right)\det U^{r+1,s+1}\left(\begin{matrix} x_i, x_j, \boldsymbol{\zeta} \\ y_i, y_j, \boldsymbol{\omega}\end{matrix}\,\middle|\,\begin{matrix} c_i, c_j, \boldsymbol{\gamma} \\ d_i, d_j, \boldsymbol{\delta}\end{matrix}\right)}{\det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}}\right)_{1\leqslant i<j\leqslant 2n}.$$

On the other hand, using the relations (3.1) and

$$\prod_{1\leqslant i<j\leqslant 2n}\left(\frac{y_j}{x_j} - \frac{y_i}{x_i}\right) = \prod_{k=1}^{2n} x_k^{-2n+1}\prod_{1\leqslant i<j\leqslant 2n}\det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix},$$

we see that the right-hand side of (1.4) becomes

$$\begin{aligned}
&\frac{1}{(\prod_{k=1}^{2n} x_k)^{-2n+1}} \cdot \frac{1}{\prod_{1\leqslant i<j\leqslant 2n} \det\binom{x_i\ x_j}{y_i\ y_j}} \\
&\quad\times \left(\prod_{i=1}^{p+q} \alpha_i \xi_i^{p-1}\right)^{-n+1} \left(\prod_{i=1}^{r+s} \gamma_i \zeta_i^{r-1}\right)^{-n+1} \det U^{p,q}\left(\begin{array}{c|c} \boldsymbol{\xi} & \boldsymbol{\alpha} \\ \boldsymbol{\eta} & \boldsymbol{\beta} \end{array}\right)^{n-1} \det U^{r,s}\left(\begin{array}{c|c} \boldsymbol{\zeta} & \boldsymbol{\gamma} \\ \boldsymbol{\omega} & \boldsymbol{\delta} \end{array}\right)^{n-1} \\
&\quad\times \left(\prod_{i=1}^{2n} a_i x_i^{n+p-1} \prod_{i=1}^{p+q} \alpha_i \xi_i^{n+p-1}\right)^{-1} \left(\prod_{i=1}^{2n} c_i x_i^{n+r-1} \prod_{i=1}^{r+s} \gamma_i \zeta_i^{n+r-1}\right)^{-1} \\
&\quad\times \det U^{n+p,n+q}\left(\begin{array}{c|c} \boldsymbol{x}, \boldsymbol{\xi} & \boldsymbol{a}, \boldsymbol{\alpha} \\ \boldsymbol{y}, \boldsymbol{\eta} & \boldsymbol{b}, \boldsymbol{\beta} \end{array}\right) \det U^{n+r,n+s}\left(\begin{array}{c|c} \boldsymbol{x}, \boldsymbol{\zeta} & \boldsymbol{c}, \boldsymbol{\gamma} \\ \boldsymbol{y}, \boldsymbol{\omega} & \boldsymbol{d}, \boldsymbol{\delta} \end{array}\right).
\end{aligned}$$

Comparing the both sides and canceling the common factors, we obtain the desired identity (3.3). □

In this setting, a homogeneous version of (1.3) is a direct consequence of (3.3). A key is the following relation between determinant and Pfaffian. If A is any $m \times (2n-m)$ matrix, then we have

$$\operatorname{Pf}\begin{pmatrix} O & A \\ -{}^tA & O \end{pmatrix} = \begin{cases} (-1)^{n(n-1)/2} \det A & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \tag{3.5}$$

Corollary 3.3. *Let n be a positive integer and let p and q be fixed nonnegative integers. For vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, $\boldsymbol{w}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ of length n, and vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ of length $p+q$, we have*

$$\begin{aligned}
&\det\left(\frac{\det U^{p+1,q+1}\left(\begin{array}{c|c} x_i, z_j, \boldsymbol{\xi} & a_i, c_j, \boldsymbol{\alpha} \\ y_i, w_j, \boldsymbol{\eta} & b_i, d_j, \boldsymbol{\beta} \end{array}\right)}{\det\binom{x_i\ z_j}{y_i\ w_j}}\right)_{1\leqslant i,j\leqslant n} = \frac{(-1)^{n(n-1)/2}}{\prod_{1\leqslant i,j\leqslant n} \det\binom{x_i\ z_j}{y_i\ w_j}} \\
&\quad\times \det U^{p,q}\left(\begin{array}{c|c} \boldsymbol{\xi} & \boldsymbol{\alpha} \\ \boldsymbol{\eta} & \boldsymbol{\beta} \end{array}\right)^{n-1} \det U^{n+p,n+q}\left(\begin{array}{c|c} \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\xi} & \boldsymbol{a}, \boldsymbol{c}, \boldsymbol{\alpha} \\ \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{\eta} & \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{\beta} \end{array}\right).
\end{aligned} \tag{3.6}$$

Proof. In (3.3), we take $r = s = 0$ and put

$$\begin{aligned}
c_1 &= \cdots = c_n = 1, \qquad c_{n+1} = \cdots = c_{2n} = 0, \\
d_1 &= \cdots = d_n = 0, \qquad d_{n+1} = \cdots = d_{2n} = 1.
\end{aligned} \tag{3.7}$$

Under this substitution (3.7), we have

$$\det U^{1,1}\left(\begin{array}{c|c} x_i, x_j & c_i, c_j \\ y_i, y_j & d_i, d_j \end{array}\right) = \det\begin{pmatrix} c_i & d_i \\ c_j & d_j \end{pmatrix} = \begin{cases} 0 & \text{if } 1 \leqslant i, j \leqslant n \text{ or } n+1 \leqslant i, j \leqslant 2n, \\ 1 & \text{if } 1 \leqslant i \leqslant n \text{ and } n+1 \leqslant j \leqslant 2n, \\ -1 & \text{if } n+1 \leqslant i \leqslant 2n \text{ and } 1 \leqslant j \leqslant n. \end{cases}$$

Hence, by (3.5), we see that the left-hand side of (3.3) becomes

$$(-1)^{n(n-1)/2}\det\left(\frac{\det U^{p+1,q+1}\begin{pmatrix} x_i, x_{n+j}, \boldsymbol{\xi} & a_i, a_{n+j}, \boldsymbol{\alpha} \\ y_i, y_{n+j}, \boldsymbol{\eta} & b_i, b_{n+j}, \boldsymbol{\beta}\end{pmatrix}}{\det\begin{pmatrix} x_i & x_{n+j} \\ y_i & y_{n+j}\end{pmatrix}}\right)_{1\leqslant i,j\leqslant n}.$$

On the other hand, under the specialization (3.7), we have

$$\begin{aligned}\det U^{n,n}\left(\begin{array}{c|c} \boldsymbol{x} & \boldsymbol{c} \\ \boldsymbol{y} & \boldsymbol{d}\end{array}\right) &= \det\begin{pmatrix} \left(x_i^{n-j}y_i^{j-1}\right)_{1\leqslant i,j\leqslant n} & O \\ O & \left(x_{n+i}^{n-j}y_{n+i}^{j-1}\right)_{1\leqslant i,j\leqslant n}\end{pmatrix} \\ &= \prod_{1\leqslant i<j\leqslant n}\det\begin{pmatrix} x_i & x_j \\ y_i & y_j\end{pmatrix}\prod_{n+1\leqslant i<j\leqslant 2n}\det\begin{pmatrix} x_i & x_j \\ y_i & y_j\end{pmatrix}.\end{aligned}$$

Thus the right-hand side of (3.3) becomes

$$\frac{1}{\prod_{1\leqslant i,j\leqslant n}\det\begin{pmatrix} x_i & x_{n+j} \\ y_i & y_{n+j}\end{pmatrix}}\det U^{p,q}\left(\begin{array}{c|c} \boldsymbol{\xi} & \boldsymbol{\alpha} \\ \boldsymbol{\eta} & \boldsymbol{\beta}\end{array}\right)^{n-1}\det U^{n+p,n+q}\left(\begin{array}{c|c} \boldsymbol{x},\boldsymbol{\xi} & \boldsymbol{a},\boldsymbol{\alpha} \\ \boldsymbol{y},\boldsymbol{\eta} & \boldsymbol{b},\boldsymbol{\beta}\end{array}\right).$$

Lastly, if we replace the variables as $x_{n+i}=z_i$, $y_{n+i}=w_i$, $a_{n+i}=c_i$ and $b_{n+i}=d_i$ for $1\leqslant i\leqslant n$, then we obtain the desired identity (3.6). This completes our proof. □

Now it is easy to derive (1.3) from (3.5) by using the relation (3.1). To prove the remaining identities (1.5) and (1.6) in Theorem 1.1, we need the following lemma.

Lemma 3.4. *Let n be a nonnegative integer.*

(1) *For vectors $\boldsymbol{x}$, $\boldsymbol{a}$ and $\mathbf{1}=(1,\dots,1)$ of length $2n$, we have*

$$\det U^{n,n}\left(\begin{array}{c|c} \boldsymbol{x} & \mathbf{1}+\boldsymbol{a}\boldsymbol{x} \\ \mathbf{1}+\boldsymbol{x}^2 & \boldsymbol{x}+\boldsymbol{a}\end{array}\right)=(-1)^{n(n-1)/2}\det W^{2n}(\boldsymbol{x};\boldsymbol{a}). \tag{3.8}$$

(2) *For vectors $\boldsymbol{x}$, $\boldsymbol{a}$ and $\mathbf{1}=(1,\dots,1)$ of length $2n+1$, we have*

$$\det U^{n,n+1}\left(\begin{array}{c|c} \boldsymbol{x} & \mathbf{1}+\boldsymbol{a}\boldsymbol{x}^2 \\ \mathbf{1}+\boldsymbol{x}^2 & \mathbf{1}+\boldsymbol{a}\end{array}\right)=(-1)^{n(n-1)/2}\det W^{2n+1}(\boldsymbol{x};\boldsymbol{a}). \tag{3.9}$$

Proof. By definition, we have

$$\begin{aligned}&\det U^{p,q}\left(\begin{array}{c|c} \boldsymbol{x} & \boldsymbol{c} \\ \mathbf{1}+\boldsymbol{x}^2 & \boldsymbol{d}\end{array}\right) \\ &\quad=\det\left(\begin{cases} c_i x_i^{p-j}\left(1+x_i^2\right)^{j-1} & \text{if } 1\leqslant i\leqslant p+q \text{ and } 1\leqslant j\leqslant p, \\ d_i x_i^{p+q-j}\left(1+x_i^2\right)^{j-p-1} & \text{if } 1\leqslant i\leqslant p+q \text{ and } p+1\leqslant j\leqslant p+q.\end{cases}\right).\end{aligned}$$

By performing appropriate elementary column transformations, one can see that this determinant is equal to

$$\det\left(\begin{cases} c_i x_i^{p-1} & \text{if } 1 \leqslant i \leqslant p+q \text{ and } j=1,\\ c_i x_i^{p-j}\left(1+x_i^{2(j-1)}\right) & \text{if } 1 \leqslant i \leqslant p+q \text{ and } 2 \leqslant j \leqslant p,\\ d_i x_i^{q-1} & \text{if } 1 \leqslant i \leqslant p+q \text{ and } j=p+1,\\ d_i x_i^{p+q-j}\left(1+x_i^{2(j-p-1)}\right) & \text{if } 1 \leqslant i \leqslant p+q \text{ and } p+2 \leqslant j \leqslant p+q. \end{cases}\right). \tag{3.10}$$

(1) First put $q=p=n$, $c_i=1+a_i x_i$ and $d_i=x_i+a_i$ for $1 \leqslant i \leqslant 2n$. Then the above determinant (3.10) is equal to

$$\det\left(\begin{cases} x_i^{n-1}+a_i x_i^{n} & \text{if } 1 \leqslant i \leqslant 2n \text{ and } j=1,\\ x_i^{n-j}+a_i x_i^{n+j-1}+x_i^{n+j-2}+a_i x_i^{n-j+1} & \text{if } 1 \leqslant i \leqslant 2n \text{ and } 2 \leqslant j \leqslant n,\\ x_i^{n}+a_i x_i^{n-1} & \text{if } 1 \leqslant i \leqslant 2n \text{ and } j=n+1,\\ x_i^{2n-j+1}+a_i x_i^{j-2}+x_i^{j-1}+a_i x_i^{2n-j} & \text{if } 1 \leqslant i \leqslant 2n \text{ and } n+2 \leqslant j \leqslant 2n. \end{cases}\right).$$

Subtract the first column from the $(n+2)$th column and subtract the $(n+1)$th column from the second column, then subtract the second column from the $(n+3)$th column and subtract the $(n+2)$th column from the third column, and so on. We continue these elementary column transformations until we obtain

$$\det\left(\begin{cases} x_i^{n-j}+a_i x_i^{n+j-1} & \text{if } 1 \leqslant i \leqslant 2n \text{ and } 1 \leqslant j \leqslant n,\\ x_i^{j-1}+a_i x_i^{2n-j} & \text{if } 1 \leqslant i \leqslant 2n \text{ and } n+1 \leqslant j \leqslant 2n. \end{cases}\right),$$

which is $(-1)^{n(n-1)/2}\det W^{2n}(\boldsymbol{x};\boldsymbol{a})$.

(2) Next we take $p=n$, $q=n+1$, $c_i=1+a_i x_i^2$ and $d_i=1+a_i$ $(1 \leqslant i \leqslant 2n+1)$ in (3.10), then we obtain

$$\det\left(\begin{cases} x_i^{n-1}+a_i x_i^{n+1} & \text{if } 1 \leqslant i \leqslant 2n+1 \text{ and } j=1,\\ x_i^{n-j}+a_i x_i^{n+j}+x_i^{n+j-2}+a_i x_i^{n-j+2} & \text{if } 1 \leqslant i \leqslant 2n+1 \text{ and } 2 \leqslant j \leqslant n,\\ x_i^{n}+a_i x_i^{n} & \text{if } 1 \leqslant i \leqslant 2n+1 \text{ and } j=n+1,\\ x_i^{2n+1-j}+a_i x_i^{j-1}+x_i^{j-1}+a_i x_i^{2n+1-j} & \text{if } 1 \leqslant i \leqslant 2n+1 \text{ and } n+2 \leqslant j \leqslant 2n+1. \end{cases}\right).$$

We subtract the first column from the $(n+2)$th column and subtract the $(n+1)$th column from the second column, and then subtract the second column from the $(n+3)$th column and subtract the $(n+2)$th column from the third column, and so on. We continue these elementary transformations until we obtain

$$\det\left(\begin{cases} x_i^{n-i}+a_i x_i^{n+j} & \text{if } 1 \leqslant i \leqslant 2n+1 \text{ and } 1 \leqslant j \leqslant n,\\ x_i^{j-1}+a_i x_i^{2n+1-j} & \text{if } 1 \leqslant i \leqslant 2n+1 \text{ and } n+1 \leqslant j \leqslant 2n+1. \end{cases}\right),$$

which is equal to $(-1)^{n(n-1)/2}\det W^{2n+1}(\boldsymbol{x};\boldsymbol{a})$. This completes our proof. □

Now we can finish our proof of Theorem 1.1.

Proof of the identities (1.3),(1.5) and (1.6) in Theorem 1.1. As we mentioned before, the identity (1.3) follows from (3.6) by virtue of (3.1).

We derive (1.6) from (3.3). First we consider the case where both $p = 2l$ and $q = 2m$ are even. In (3.3), we take $p = q = l$ and $r = s = m$, and perform the following substitutions:

$$\boldsymbol{x} \leftarrow \boldsymbol{x}, \quad \boldsymbol{y} \leftarrow \mathbf{1} + \boldsymbol{x}^2, \quad \boldsymbol{\xi} \leftarrow z, \quad \boldsymbol{\eta} \leftarrow \mathbf{1} + z^2, \quad \boldsymbol{\zeta} \leftarrow \boldsymbol{w}, \quad \boldsymbol{\omega} \leftarrow \mathbf{1} + \boldsymbol{w}^2,$$
$$\boldsymbol{a} \leftarrow \mathbf{1} + \boldsymbol{a}\boldsymbol{x}, \quad \boldsymbol{b} \leftarrow \boldsymbol{x} + \boldsymbol{a}, \quad \boldsymbol{c} \leftarrow \mathbf{1} + \boldsymbol{b}\boldsymbol{x}, \quad \boldsymbol{d} \leftarrow \boldsymbol{x} + \boldsymbol{b},$$
$$\boldsymbol{\alpha} \leftarrow \mathbf{1} + \boldsymbol{c}z, \quad \boldsymbol{\beta} \leftarrow z + \boldsymbol{c}, \quad \boldsymbol{\gamma} \leftarrow \mathbf{1} + \boldsymbol{d}\boldsymbol{w}, \quad \boldsymbol{\delta} \leftarrow \boldsymbol{w} + \boldsymbol{d}.$$

By using the relation

$$\det\begin{pmatrix} x_i & x_j \\ 1 + x_i^2 & 1 + x_j^2 \end{pmatrix} = (x_i - x_j)(1 - x_i x_j),$$

and (3.8) in Lemma 3.4, we see that the identity (3.3) becomes

$$\begin{aligned}
&\mathrm{Pf}\left(\frac{(-1)^{l(l+1)/2+m(m+1)/2} \det W^{2l+2}(x_i, x_j, z; a_i, a_j, \boldsymbol{c}) \det W^{2m+2}(x_i, x_j, \boldsymbol{w}; b_i, b_j, \boldsymbol{d})}{(x_i - x_j)(1 - x_i x_j)} \right)_{1 \leqslant i < j \leqslant 2n} \\
&\quad = (-1)^{(n-1)\binom{l}{2} + (n-1)\binom{m}{2} + \binom{n+l}{2} + \binom{n+m}{2}} \frac{1}{\prod_{1 \leqslant i < j \leqslant 2n} (x_i - x_j)(1 - x_i x_j)} \\
&\qquad \times \det W^{2l}(z; \boldsymbol{c})^{n-1} \det W^{2m}(\boldsymbol{w}; \boldsymbol{d})^{n-1} \det W^{2n+2l}(\boldsymbol{x}, z; \boldsymbol{a}, \boldsymbol{c}) \det W^{2n+2m}(\boldsymbol{x}, \boldsymbol{w}; \boldsymbol{b}, \boldsymbol{d}).
\end{aligned}$$

Since we have

$$\begin{aligned}
&(n-1)l(l-1)/2 + (n-1)m(m-1)/2 + (n+l)(n+l-1)/2 + (n+m)(n+m-1)/2 \\
&\quad = n\bigl\{l(l+1)/2 + m(m+1)/2\bigr\} + n(n-1),
\end{aligned}$$

we obtain the identity (1.6) when p and q are both even. We can prove the other cases similarly by using (3.8) or (3.9) according as p and q are even or odd.

Also the remaining identity (1.5) can be derived from (3.6) by using (3.8) or (3.9). The details are left to the reader. □

4. A variation of the determinant and Pfaffian identities

In this section, we give a variation of the identities in Theorem 1.1, which can be regarded as a generalization of an identity of T. Sundquist [19]. This variation is proposed by one of the authors.

Let n be a positive integer and let p and q be nonnegative integers with $p+q=n$. Let $\boldsymbol{x}=(x_1,\dots,x_n)$ and $\boldsymbol{a}=(a_1,\dots,a_n)$ be vectors of variables. For partitions λ and μ with $l(\lambda)\leqslant p$ and $l(\mu)\leqslant q$, we define a matrix $V^{p,q}_{\lambda,\mu}(\boldsymbol{x};\boldsymbol{a})$ to be the $n\times n$ matrix with ith row

$$\left(x_i^{\lambda_p},x_i^{\lambda_{p-1}+1},x_i^{\lambda_{p-2}+2},\dots,x_i^{\lambda_1+p-1},a_ix_i^{\mu_q},a_ix_i^{\mu_{q-1}+1},a_ix_i^{\mu_{q-2}+2},\dots,a_ix_i^{\mu_1+q-1}\right).$$

For example, if $\lambda=\mu=\emptyset$, then we have $V^{p,q}_{\emptyset,\emptyset}(\boldsymbol{x};\boldsymbol{a})=V^{p,q}(\boldsymbol{x};\boldsymbol{a})$. Let $\mathcal{P}_n$ denote the set of integer partitions of the form $(\alpha_1,\dots,\alpha_r|\alpha_1+1,\dots,\alpha_r+1)$ in the Frobenius notation with $\alpha_1+2\leqslant n$. (So $n\geqslant 2$). We define

$$F^{p,q}(\boldsymbol{x};\boldsymbol{a})=\sum_{\lambda\in\mathcal{P}_p,\,\mu\in\mathcal{P}_q}(-1)^{(|\lambda|+|\mu|)/2}\det V^{p,q}_{\lambda,\mu}(\boldsymbol{x};\boldsymbol{a}).$$

For example, if $p=q=1$, then $F^{1,1}(\boldsymbol{x};\boldsymbol{a})=a_2-a_1$, and, if $p=q=2$, then $\mathcal{P}_2=\{\emptyset,(1,1)\}$ and

$$\begin{aligned}F^{2,2}(\boldsymbol{x};\boldsymbol{a})&=\det\begin{pmatrix}1&x_1&a_1&a_1x_1\\1&x_2&a_2&a_2x_2\\1&x_3&a_3&a_3x_3\\1&x_4&a_4&a_4x_4\end{pmatrix}-\det\begin{pmatrix}x_1&x_1^2&a_1&a_1x_1\\x_2&x_2^2&a_2&a_2x_2\\x_3&x_3^2&a_3&a_3x_3\\x_4&x_4^2&a_4&a_4x_4\end{pmatrix}\\&\quad-\det\begin{pmatrix}1&x_1&a_1x_1&a_1x_1^2\\1&x_2&a_2x_2&a_2x_2^2\\1&x_3&a_3x_3&a_3x_3^2\\1&x_4&a_4x_4&a_4x_4^2\end{pmatrix}+\det\begin{pmatrix}x_1&x_1^2&a_1x_1&a_1x_1^2\\x_2&x_2^2&a_2x_2&a_2x_2^2\\x_3&x_3^2&a_3x_3&a_3x_3^2\\x_4&x_4^2&a_4x_4&a_4x_4^2\end{pmatrix}.\end{aligned}$$

The aim of this section is to prove the following theorem:

Theorem 4.1.

(a) *Let n be a positive integer and let p and q be nonnegative integers. For six vectors of variables*

$$\boldsymbol{x}=(x_1,\dots,x_n),\quad \boldsymbol{y}=(y_1,\dots,y_n),\quad \boldsymbol{z}=(z_1,\dots,z_{p+q}),$$
$$\boldsymbol{a}=(a_1,\dots,a_n),\quad \boldsymbol{b}=(b_1,\dots,b_n),\quad \boldsymbol{c}=(c_1,\dots,c_{p+q})$$

we have

$$\begin{aligned}&\det\left(\frac{F^{p+1,q+1}(x_i,y_j,\boldsymbol{z};a_i,b_j,\boldsymbol{c})}{(y_j-x_i)(1-x_iy_j)}\right)_{1\leqslant i,j\leqslant n}\\&\quad=\frac{(-1)^{n(n-1)/2}}{\prod_{i,j=1}^n(y_j-x_i)(1-x_iy_j)}F^{p,q}(\boldsymbol{z};\boldsymbol{c})^{n-1}F^{n+p,n+q}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z};\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}).\end{aligned}\tag{4.1}$$

(b) *Let n be a positive integer and let p, q, r, s be nonnegative integers. For seven vectors of variables*

$$\boldsymbol{x} = (x_1, \dots, x_{2n}), \quad \boldsymbol{a} = (a_1, \dots, a_{2n}), \quad \boldsymbol{b} = (b_1, \dots, b_{2n}),$$
$$\boldsymbol{z} = (z_1, \dots, z_{p+q}), \quad \boldsymbol{c} = (c_1, \dots, c_{p+q}),$$
$$\boldsymbol{w} = (w_1, \dots, w_{r+s}), \quad \boldsymbol{d} = (d_1, \dots, d_{r+s}),$$

we have

$$\begin{aligned}
&\operatorname{Pf}\left(\frac{F^{p+1,q+1}(x_i, x_j, \boldsymbol{z}; a_i, a_j, \boldsymbol{c}) F^{r+1,s+1}(x_i, x_j, \boldsymbol{w}; b_i, b_j, \boldsymbol{d})}{(x_j - x_i)(1 - x_i x_j)}\right)_{1 \leqslant i,j \leqslant 2n} \\
&\quad = \frac{1}{\prod_{1 \leqslant i < j \leqslant 2n}(x_j - x_i)(1 - x_i x_j)} F^{p,q}(\boldsymbol{z}; \boldsymbol{c})^{n-1} F^{r,s}(\boldsymbol{w}; \boldsymbol{d})^{n-1} \\
&\qquad \times F^{n+p,n+q}(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{a}, \boldsymbol{c}) F^{n+r,n+s}(\boldsymbol{x}, \boldsymbol{w}; \boldsymbol{b}, \boldsymbol{d}).
\end{aligned} \tag{4.2}$$

In particular, by putting $p = q = r = s = 0$ and $b_i = x_i$ for $1 \leqslant i \leqslant 2n$ in (4.2), we obtain Sundquist's identity [19, Theorem 2.1].

Corollary 4.2. (*Sundquist*)

$$\begin{aligned}
&\operatorname{Pf}\left(\frac{a_j - a_i}{1 - x_i x_j}\right)_{1 \leqslant i < j \leqslant 2n} \\
&\quad = \frac{(-1)^{n(n-1)/2}}{\prod_{1 \leqslant i < j \leqslant 2n}(1 - x_i x_j)} \sum_{\lambda, \mu \in \mathcal{P}_n} (-1)^{(|\lambda| + |\mu|)/2} \det V_{\lambda,\mu}^{n,n}(\boldsymbol{x}; \boldsymbol{a}).
\end{aligned} \tag{4.3}$$

In order to prove Theorem 4.1 and Corollary 4.2, we need a relation between $F^{p,q}(\boldsymbol{x}; \boldsymbol{a})$ and $\det V^{p,q}(\boldsymbol{y}; \boldsymbol{b})$.

Proposition 4.3. *We have*

$$\begin{aligned}
F^{p,q}(\boldsymbol{x}; \boldsymbol{a}) &= (-1)^{\binom{p}{2} + \binom{q}{2}} \prod_{i=1}^{p+q} x_i^{p-1} \cdot \det V^{p,q}\big(\boldsymbol{x} + \boldsymbol{x}^{-1}; \boldsymbol{a}\boldsymbol{x}^{q-p}\big), \\
&= (-1)^{\binom{p}{2} + \binom{q}{2}} \det U^{p,q}\left(\begin{array}{c|c} \boldsymbol{x} & \mathbf{1} \\ \mathbf{1} + \boldsymbol{x}^2 & \boldsymbol{a} \end{array}\right),
\end{aligned} \tag{4.4}$$

where

$$\boldsymbol{x} + \boldsymbol{x}^{-1} = \big(x_1 + x_1^{-1}, \dots, x_{p+q} + x_{p+q}^{-1}\big),$$
$$\boldsymbol{a}\boldsymbol{x}^{q-p} = \big(a_1 x_1^{q-p}, \dots, a_{p+q} x_{p+q}^{q-p}\big),$$
$$\mathbf{1} + \boldsymbol{x}^2 = \big(1 + x_1^2, \dots, 1 + x_{p+q}^2\big).$$

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Here we give a proof by using the Cauchy–Binet formula. Let $A = (a_{ij})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$ be an m by n matrix. For any subsets $I = \{i_1 < \cdots < i_r\} \subset [m]$, and $J = \{j_1 < \cdots < j_r\} \subset [n]$, let $\Delta^I_J(A)$ denote the submatrix obtained by selecting the rows indexed by I and the columns indexed by J. If all rows or columns are selected, i.e., if $I = [m]$ or $J = [n]$, then we simply write $\Delta_J(A)$ or $\Delta^I(A)$ for $\Delta^{[m]}_J(A)$ or $\Delta^I_{[n]}(A)$.

Lemma 4.4. *Let X and Y be any $n \times N$ matrix and A be any $N \times N$ matrix. Then we have*

$$\det\bigl(XA^tY\bigr) = \sum_{I,J} \det \Delta^I_J(A) \det \Delta_I(X) \det \Delta_J(Y), \tag{4.5}$$

where the sum is taken over all pairs (I, J) of n-element subsets of $[N]$.

For a partition λ with length $\leqslant r$, we put

$$I(\lambda) = \{\lambda_r, \lambda_{r-1} + 1, \lambda_{r-2} + 2, \ldots, \lambda_1 + r - 1\}.$$

Then a key of the proof of Proposition 4.3 is the following lemma.

Lemma 4.5. *Let D_r be the following $r \times (2r - 1)$ matrix with columns indexed by $0, 1, \ldots, 2r - 2$:*

$$D_r = \begin{array}{c} \begin{matrix} 0 & & r-2 & r-1 & r & & 2r-2 \end{matrix} \\ \begin{pmatrix} & & & 1 & & & \\ & & 1 & & 1 & & \\ & \iddots & & & & \ddots & \\ 1 & & & & & & 1 \end{pmatrix} \end{array}.$$

Then the minor of D_r corresponding to a partition λ is given by

$$\det \Delta_{I(\lambda)}(D_r) = \begin{cases} (-1)^{r(r-1)/2+|\lambda|/2} & \textit{if } \lambda \in \mathcal{P}_r, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. First we show that $\det \Delta_{I(\lambda)}(D_r) = 0$ unless $\lambda \in \mathcal{P}_r$. Suppose $\det \Delta_{I(\lambda)}(D_r) \neq 0$. Since the first row of the matrix D_r has only 1 in the $(r - 1)$th column, we must have $r - 1 \in I(\lambda)$. If we denote by $p = p(\lambda)$ the length of the main diagonal of the Young diagram of λ, then we have

$$p = \#\{i\colon \lambda_i \geqslant i\} = \#\{i\colon \lambda_i + r - i \geqslant r\} = \#\{k \in I(\lambda)\colon k \geqslant r\}.$$

Hence the elements $\lambda_1 + r - 1 > \cdots > \lambda_p + r - p$ are the largest p elements belonging in $I(\lambda)$. Since the Frobenius' Lemma ([12, (1.7)]) says that

$$\{\lambda_i + r - i\colon 1 \leqslant i \leqslant r\} \cup \bigl\{r - 1 + j - {}^t\lambda_j\colon 1 \leqslant j \leqslant r - 1\bigr\} = \{0, 1, \ldots, 2r - 2\},$$

we see that the elements $r - {}^t\lambda_1 < \cdots < r - 1 + p - {}^t\lambda_p$ are the smallest p elements not belonging in $I(\lambda)$. On the other hand, by noting that the kth column of D_r is identical with the $(2r-2-k)$th column, we have, if $k \neq r-1$, then $k \in I(\lambda)$ if and only if $2r-2-k \notin I(\lambda)$. Therefore we have

$$\lambda_i + r - i + \bigl(r - 1 + i - {}^t\lambda_i\bigr) = 2r - 2 \quad (1 \leqslant i \leqslant p).$$

So we have ${}^t\lambda_i = \lambda_i + 1$ for $1 \leqslant i \leqslant p$, which implies $\lambda \in \mathcal{P}_r$.

Next we show that, if $\lambda = (\alpha|\alpha+1) \in \mathcal{P}_r$, then $\det \Delta_{I(\lambda)}(D_r) = (-1)^{r(r-1)/2+|\lambda|/2} = (-1)^{r(r-1)/2+|\alpha|+p}$. Note that $\Delta_{I(\lambda)}(D_r)$ is a permutation matrix. Let σ be the permutation corresponding to $\Delta_{I(\lambda)}(D_r)$. Then we have

$$\sigma(1) > \sigma(2) > \cdots > \sigma(r-p-1),$$
$$\sigma(r-p) = 1, \qquad \sigma(r-p+1) = \alpha_p + 2, \qquad \ldots, \qquad \sigma(r) = \alpha_1 + 2.$$

The number of pairs (i, j) such that $i < j$ and $\sigma(i) < \sigma(j)$ is equal to

$$(\alpha_1 + 1) + \cdots + (\alpha_p + 1) = |\alpha| + p,$$

so we have

$$\det \Delta_{I(\lambda)}(D_r) = \operatorname{sgn}(\sigma) = (-1)^{r(r-1)/2-|\alpha|-p}. \qquad \square$$

Proof of Proposition 4.3. In this proof we put $m = p + q$ for brevity. Apply the Cauchy–Binet formula (4.4) to the following $(p+q) \times (2p+2q-2)$ matrices X and Y (and the identity matrix A):

$$X = \begin{pmatrix} D_p & O \\ O & D_q \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & x_1 & \cdots & x_1^{2p-2} & a_1 & a_1x_1 & \cdots & a_1x_1^{2q-2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_m & \cdots & x_m^{2p-2} & a_m & a_mx_m & \cdots & a_mx_m^{2q-2} \end{pmatrix}.$$

Let $C \cup C' = \{0, 1, \ldots, 2p-2\} \cup \{0', 1', \ldots, (2q-2)'\}$ be the set indexing the columns of X and Y. Let I be a subset of $C \cup C'$ which has cardinality $p+q$ and consider the minors of X and Y obtained by choosing the columns with indices in I. Then we have

$$\det \Delta_I(X) = 0 \quad \text{unless } \#(I \cap C) = p \text{ and } \#(I \cap C') = q.$$

Suppose that $\#(I \cap C) = p$ and $\#(I \cap C') = q$ and that the subsets $I \cap C$ and $I \cap C'$ determine partitions λ and μ, i.e., $I(\lambda) = I \cap C$ and $I(\mu) = I \cap C'$. Then, by Lemma 4.5 and the definition of $V^{p,q}_{\lambda,\mu}(\boldsymbol{x}; \boldsymbol{a})$, we have

$$\det \Delta_I(X) = \begin{cases} (-1)^{\binom{p}{2}+\binom{q}{2}+|\lambda|/2+|\mu|/2} & \text{if } \lambda \in \mathcal{P}_p \text{ and } \mu \in \mathcal{P}_q, \\ 0 & \text{otherwise,} \end{cases}$$
$$\det \Delta_I(Y) = \det V^{p,q}_{\lambda,\mu}(\boldsymbol{x}; \boldsymbol{a}).$$

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Hence the Cauchy–Binet formula gives

$$\begin{aligned}\det\bigl(X^tY\bigr) &= \sum_{I\subset C\cup C'} \det\Delta_I(X)\det\Delta_I(Y)\\ &= \sum_{\lambda\in\mathcal{P}_p,\,\mu\in\mathcal{P}_q} (-1)^{\binom{p}{2}+\binom{q}{2}+|\lambda|/2+|\mu|/2}\det V^{p,q}_{\lambda,\mu}(\boldsymbol{x};\boldsymbol{a}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\det\bigl(X^tY\bigr)\\ &= \det\begin{pmatrix} x_1^{p-1} & x_1^p+x_1^{p-2} & \cdots & x_1^{2p-2}+1 & a_1x_1^{q-1} & a_1(x_1^q+x_1^{q-2}) & \cdots & a_1(x_1^{2q-2}+1)\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ x_m^{p-1} & x_m^p+x_m^{p-2} & \cdots & x_m^{2p-2}+1 & a_mx_m^{q-1} & a_m(x_m^q+x_m^{q-2}) & \cdots & a_m(x_m^{2q-2}+1)\end{pmatrix}.\end{aligned}$$

Now, by applying elementary transformations and by using the relations

$$\bigl(x+x^{-1}\bigr)^{2k} = \sum_{i=0}^{k-1}\binom{2k}{i}\bigl(x^{2k-2i}+x^{-2k+2i}\bigr)+\binom{2k}{k},$$

$$\bigl(x+x^{-1}\bigr)^{2k+1} = \sum_{i=0}^{k}\binom{2k+1}{i}\bigl(x^{2k-2i+1}+x^{-2k+2i-1}\bigr),$$

we have

$$\begin{aligned}\det\bigl(X^tY\bigr) &= \prod_{i=1}^m x_i^{p-1}\cdot\det\left(\begin{cases}(x_i+x_i^{-1})^{j-1} & \text{if } 1\leqslant i\leqslant m \text{ and } 1\leqslant j\leqslant p,\\ a_ix_i^{q-p}(x_i+x_i^{-1})^{j-p-1} & \text{if } 1\leqslant i\leqslant m \text{ and } p+1\leqslant j\leqslant p+q.\end{cases}\right)\\ &= \prod_{i=1}^m x_i^{p-1}\cdot\det V^{p,q}\bigl(\boldsymbol{x}+\boldsymbol{x}^{-1};\boldsymbol{a}\boldsymbol{x}^{q-p}\bigr)\\ &= U^{p,q}\left(\begin{array}{c|c}\boldsymbol{x} & \boldsymbol{1}\\ \boldsymbol{1}+\boldsymbol{x}^2 & \boldsymbol{a}\end{array}\right). \qquad\square\end{aligned}$$

Remark 4.6. By the above argument in the case of $q=0$, we actually show one of the Littlewood's formula

$$\sum_{\lambda\in\mathcal{P}_n} s_\lambda(x_1,\dots,x_n) = \prod_{1\leqslant i<j\leqslant n}(1-x_ix_j).$$

Now we are in position to derive Theorem 4.1 from Theorem 3.2.

Proof of Theorem 4.1. Note that

$$\det\begin{pmatrix} x & 1+x^2 \\ y & 1+y^2 \end{pmatrix} = (x-y)(1-xy). \tag{4.6}$$

First we prove (4.2). From the above relation (4.6) and (4.4), we have

$$\begin{aligned}
&\frac{F^{p+1,q+1}(x_i,x_j,\boldsymbol{z};a_i,a_j,\boldsymbol{c})F^{r+1,s+1}(x_i,x_j,\boldsymbol{w};b_i,b_j,\boldsymbol{d})}{(x_j-x_i)(1-x_ix_j)} \\
&\quad= \frac{(-1)^{1+\binom{p+1}{2}+\binom{q+1}{2}+\binom{r+1}{2}+\binom{s+1}{2}}}{\det\begin{pmatrix} x_i & 1+x_i^2 \\ x_j & 1+x_j^2\end{pmatrix}} \\
&\qquad\times \det U^{p+1,q+1}\left(\begin{matrix} x_i,x_j,\boldsymbol{z} \\ 1+x_i^2,1+x_j^2,\boldsymbol{1}+\boldsymbol{z}^2\end{matrix}\,\middle|\,\begin{matrix} 1,1,\boldsymbol{1} \\ a_i,a_j,\boldsymbol{c}\end{matrix}\right) \\
&\qquad\times \det U^{r+1,s+1}\left(\begin{matrix} x_i,x_j,\boldsymbol{w} \\ 1+x_i^2,1+x_j^2,\boldsymbol{1}+\boldsymbol{w}^2\end{matrix}\,\middle|\,\begin{matrix} 1,1,\boldsymbol{1} \\ b_i,b_j,\boldsymbol{d}\end{matrix}\right).
\end{aligned}$$

Hence we apply (3.6) to obtain

$$\begin{aligned}
&\operatorname{Pf}\left(\frac{F^{p+1,q+1}(x_i,x_j,\boldsymbol{z};a_i,a_j,\boldsymbol{c})F^{r+1,s+1}(x_i,x_j,\boldsymbol{w};b_i,b_j,\boldsymbol{d})}{(x_j-x_i)(1-x_ix_j)}\right)_{1\leqslant i,j\leqslant 2n} \\
&\quad= \frac{(-1)^{n+n\binom{p+1}{2}+n\binom{q+1}{2}+n\binom{r+1}{2}+n\binom{s+1}{2}}}{\prod_{1\leqslant i<j\leqslant 2n}\det\begin{pmatrix} x_i & 1+x_i^2 \\ x_j & 1+x_j^2\end{pmatrix}} \\
&\qquad\times \det U^{p,q}\left(\begin{matrix}\boldsymbol{z} \\ \boldsymbol{1}+\boldsymbol{z}^2\end{matrix}\,\middle|\,\begin{matrix}\boldsymbol{1} \\ \boldsymbol{c}\end{matrix}\right)^{n-1} \det U^{r,s}\left(\begin{matrix}\boldsymbol{w} \\ \boldsymbol{1}+\boldsymbol{w}^2\end{matrix}\,\middle|\,\begin{matrix}\boldsymbol{1} \\ \boldsymbol{d}\end{matrix}\right)^{n-1} \\
&\qquad\times \det U^{n+p,n+q}\left(\begin{matrix}\boldsymbol{x},\boldsymbol{z} \\ \boldsymbol{1}+\boldsymbol{x}^2,\boldsymbol{1}+\boldsymbol{z}^2\end{matrix}\,\middle|\,\begin{matrix}\boldsymbol{1},\boldsymbol{1} \\ \boldsymbol{a},\boldsymbol{c}\end{matrix}\right) \det U^{n+r,n+s}\left(\begin{matrix}\boldsymbol{x},\boldsymbol{w} \\ \boldsymbol{1}+\boldsymbol{x}^2,\boldsymbol{1}+\boldsymbol{w}^2\end{matrix}\,\middle|\,\begin{matrix}\boldsymbol{1},\boldsymbol{1} \\ \boldsymbol{b},\boldsymbol{d}\end{matrix}\right) \\
&\quad= (-1)^{n\binom{p+1}{2}+n\binom{q+1}{2}+n\binom{r+1}{2}+n\binom{s+1}{2}+(n-1)\binom{p}{2}+(n-1)\binom{q}{2}+(n-1)\binom{r}{2}+(n-1)\binom{s}{2}+\binom{n+p}{2}+\binom{n+q}{2}+\binom{n+r}{2}+\binom{n+s}{2}} \\
&\qquad\times \frac{1}{\prod_{1\leqslant i<j\leqslant 2n}(x_j-x_i)(1-x_ix_j)}F^{p,q}(\boldsymbol{z};\boldsymbol{c})^{n-1}F^{r,s}(\boldsymbol{w};\boldsymbol{d})^{n-1} \\
&\qquad\times F^{n+p,n+q}(\boldsymbol{x},\boldsymbol{z};\boldsymbol{a},\boldsymbol{c})F^{n+r,n+s}(\boldsymbol{x},\boldsymbol{w};\boldsymbol{b},\boldsymbol{d}).
\end{aligned}$$

If we use the relation

$$n\binom{p+1}{2}-(n-1)\binom{p}{2}-\binom{p+n}{2}=-\binom{n}{2},$$

then we obtain the desired identity. This proves (4.2).

The determinant identity (4.1) can be proven by the same method by using (4.6), (4.4) and (3.3), so we omit the detailed proof. □

Proof of Corollary 4.2. If we put $p = q = r = s = 0$ and $b_i = x_i$ $(1 \leqslant i \leqslant 2n)$ in (4.2), then we have

$$\mathrm{Pf}\left(\frac{a_j - a_i}{1 - x_i x_j}\right)_{1 \leqslant i,j \leqslant 2n} = \frac{1}{\prod_{1 \leqslant i < j \leqslant 2n}(x_j - x_i)(1 - x_i x_j)} F^{n,n}(\boldsymbol{x}; \boldsymbol{a}) F^{n,n}(\boldsymbol{x}; \boldsymbol{x}).$$

From the relation (4.4), we see that

$$F^{n,n}(\boldsymbol{x}; \boldsymbol{x}) = \prod_{i=1}^{2n} x_i^{n-1} \det V^{n,n}(\boldsymbol{x} + \boldsymbol{x}^{-1}; \boldsymbol{x}).$$

And, by applying appropriate elementary column transformations, we obtain

$$F^{n,n}(\boldsymbol{x}; \boldsymbol{x}) = (-1)^{n(n-1)/2} \det\left(x_i^{j-1}\right)_{1 \leqslant i,j \leqslant 2n} = (-1)^{n(n-1)/2} \prod_{1 \leqslant i < j \leqslant 2n} (x_j - x_i).$$

This completes the proof of the corollary. □

In Theorem 4.1, we can replace $F^{p,q}(\boldsymbol{x}; \boldsymbol{a})$ by the following linear combination of $\det V^{p,q}_{\lambda,\mu}(\boldsymbol{x}; \boldsymbol{a})$:

$$G^{p,q}(\boldsymbol{x}; \boldsymbol{a}) = \sum_{\lambda \in \mathcal{Q}_p,\, \mu \in \mathcal{Q}_q} (-1)^{(|\lambda| + |\mu|)/2} \det V^{p,q}_{\lambda,\mu}(\boldsymbol{x}; \boldsymbol{a}),$$
$$H^{p,q}(\boldsymbol{x}; \boldsymbol{a}) = \sum_{\lambda \in \mathcal{R}_p,\, \mu \in \mathcal{R}_q} (-1)^{(|\lambda| + p(\lambda) + |\mu| + p(\mu))/2} \det V^{p,q}_{\lambda,\mu}(\boldsymbol{x}; \boldsymbol{a}),$$

where $\mathcal{Q}_n$ (respectively $\mathcal{R}_n$) is the set of partitions λ with length $\leqslant n$ which is of the form $\lambda = (\alpha + 1|\alpha)$ (respectively $\lambda = (\alpha|\alpha)$) in the Frobenius notation.

The following Lemma and Proposition can be proven by the same idea as Lemma 4.5 and Proposition 4.3, so we leave the proof to the reader.

Lemma 4.7. *Let C_r (respectively B_r) be the $r \times (2r+1)$ (respectively $r \times 2r$) matrix given by*

$$B_r = \begin{array}{c} \begin{array}{cccccccc} 0 & & r-2 & r-1 & r & r+1 & & 2r-1 \end{array} \\ \begin{pmatrix} & & & -1 & 1 & & & \\ & & -1 & & & 1 & & \\ & \iddots & & & & & \ddots & \\ -1 & & & & & & & 1 \end{pmatrix} \end{array},$$

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$$C_r = \begin{array}{c} \begin{matrix} 0 & & r-2 & r-1 & r & r+1 & r+2 & & 2r \end{matrix} \\ \begin{pmatrix} & & & -1 & 0 & 1 & & & \\ & & -1 & & & & 1 & & \\ & \iddots & & & & & & \ddots & \\ -1 & & & & & & & & 1 \end{pmatrix} \end{array}.$$

Then we have

(1) *For a partition* λ *of length* $\leqslant r$, *we have*

$$\det \Delta_I(\lambda)(B_r) = \begin{cases} (-1)^{\binom{r+1}{2}+(|\lambda|+p(\lambda))/2} & \text{if } \lambda \in \mathcal{R}_r, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *For a partition* λ *of length* $\leqslant r$, *we have*

$$\det \Delta_I(\lambda)(C_r) = \begin{cases} (-1)^{\binom{r+1}{2}+|\lambda|/2} & \text{if } \lambda \in \mathcal{Q}_r, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.8.

$$\begin{aligned} G^{p,q}(\boldsymbol{x};\boldsymbol{a}) &= (-1)^{\binom{p}{2}+\binom{q}{2}} \prod_{i=1}^{p+q} x_i^{p-1}\bigl(1-x_i^2\bigr) \cdot \det V^{p,q}\bigl(\boldsymbol{x}+\boldsymbol{x}^{-1};\boldsymbol{a}\boldsymbol{x}^{q-p}\bigr), \\ &= (-1)^{\binom{p}{2}+\binom{q}{2}} \prod_{i=1}^{p+q} \bigl(1-x_i^2\bigr) \cdot \det U^{p,q} \left(\begin{array}{c|c} \boldsymbol{x} & \boldsymbol{1} \\ \boldsymbol{1}+\boldsymbol{x}^2 & \boldsymbol{a} \end{array}\right), \\ H^{p,q}(\boldsymbol{x};\boldsymbol{a}) &= (-1)^{\binom{p}{2}+\binom{q}{2}} \prod_{i=1}^{p+q} x_i^{p-1}(1-x_i) \cdot \det V^{p,q}\bigl(\boldsymbol{x}+\boldsymbol{x}^{-1};\boldsymbol{a}\boldsymbol{x}^{q-p}\bigr), \\ &= (-1)^{\binom{p}{2}+\binom{q}{2}} \prod_{i=1}^{p+q} (1-x_i) \cdot \det U^{p,q} \left(\begin{array}{c|c} \boldsymbol{x} & \boldsymbol{1} \\ \boldsymbol{1}+\boldsymbol{x}^2 & \boldsymbol{a} \end{array}\right). \end{aligned}$$

In particular, we have

$$\begin{aligned} G^{p,q}(\boldsymbol{x};\boldsymbol{a}) &= \prod_{i=1}^{p+q} \bigl(1-x_i^2\bigr) \cdot F^{p,q}(\boldsymbol{x};\boldsymbol{a}), \\ H^{p,q}(\boldsymbol{x};\boldsymbol{a}) &= \prod_{i=1}^{p+q} (1-x_i) \cdot F^{p,q}(\boldsymbol{x};\boldsymbol{a}). \end{aligned}$$

From these relations, we have determinant and Pfaffian identities involving $G^{p,q}(\boldsymbol{x};\boldsymbol{a})$ and $H^{p,q}(\boldsymbol{x};\boldsymbol{a})$ similar to (4.1) and (4.2). More generally, we can consider, for example,

$$\sum_{\lambda\in\mathcal{P}_p,\,\mu\in\mathcal{Q}_q}(-1)^{(|\lambda|+|\mu|)/2}\det V^{p,q}_{\lambda,\mu}(\boldsymbol{x};\boldsymbol{a}),$$

which can be expressed in terms of $\det V^{p,q}$ or $\det U^{p,q}$.

5. Another generalization of Cauchy's determinant identity

In this section, we give another type of generalized Cauchy's determinant identities involving $\det V^{p,q}$ and $\det W^{p}$.

Theorem 5.1.

$$\det\left(\frac{1}{\det V^{p+1,q+1}(x_i,y_j,\boldsymbol{z};a_i,b_j,\boldsymbol{c})}\right)_{1\leqslant i,j\leqslant n}$$
$$=\frac{(-1)^{n(n-1)/2}\prod_{1\leqslant i<j\leqslant n}\det V^{p+1,q+1}(x_i,x_j,\boldsymbol{z};a_i,a_j,\boldsymbol{c})\det V^{p+1,q+1}(y_i,y_j,\boldsymbol{z};b_i,b_j,\boldsymbol{c})}{\prod_{i,j=1}^{n}\det V^{p+1,q+1}(x_i,y_j,\boldsymbol{z};a_i,b_j,\boldsymbol{c})},\quad(5.1)$$

$$\det\left(\frac{1}{\det W^{p+2}(x_i,y_j,\boldsymbol{z};a_i,b_j,\boldsymbol{c})}\right)_{1\leqslant i,j\leqslant n}$$
$$=\frac{(-1)^{n(n-1)/2}\prod_{1\leqslant i<j\leqslant n}\det W^{p+2}(x_i,x_j,\boldsymbol{z};a_i,a_j,\boldsymbol{c})\det W^{p+2}(y_i,y_j,\boldsymbol{z};b_i,b_j,\boldsymbol{c})}{\prod_{i,j=1}^{n}\det W^{p+2}(x_i,y_j,\boldsymbol{z};a_i,b_j,\boldsymbol{c})}.\quad(5.2)$$

If $p=q=0$, then the identity (5.1) becomes

$$\det\left(\frac{1}{b_j-a_i}\right)_{1\leqslant i,j\leqslant n}=\frac{(-1)^{n(n-1)/2}\prod_{1\leqslant i<j\leqslant n}(a_j-a_i)(b_j-b_i)}{\prod_{i,j=1}^{n}(b_j-a_i)},$$

which is equivalent to Cauchy's determinant identity (1.1).

In order to prove Theorem 5.1, we put

$$f(x,y;a,b)=\det V^{p+1,q+1}(x,y,\boldsymbol{z};a,b,\boldsymbol{c}),\quad\text{or}\quad\det W^{p+2}(x,y,\boldsymbol{z};a,b,\boldsymbol{c}).$$

The proof is based on the quadratic relations among $f(x,y;a,b)$'s, which follows from the Plücker relation for determinants.

Lemma 5.2. *Let* $(a_1,a_2,a_3,a_4,x_1,\dots,x_m)$ *be a* $(m+2)\times(m+4)$ *matrix. If we put*

$$D(i,j)=\det(a_i,a_j,x_1,\dots,x_m),$$

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then we have

$$D(1,2)D(3,4) - D(1,3)D(2,4) + D(1,4)D(2,3) = 0.$$

Proposition 5.3. *The polynomials* $f(x,y;a,b)$*'s satisfy*

$$\begin{aligned} &f(x_1,x_2;a_1,a_2)f(y_1,y_2;b_1,b_2) - f(x_1,y_1;a_1,b_1)f(x_2,y_2;a_2,b_2) \\ &\quad + f(x_1,y_2;a_1,b_2)f(x_2,y_1;a_2,b_1) = 0. \end{aligned} \tag{5.3}$$

Proof. Apply the Plücker relation to the transposes of the matrices

$$\begin{pmatrix} 1 & x_1 & \cdots & a_1x_1^q \\ 1 & x_2 & \cdots & a_2x_2^q \\ 1 & y_1 & \cdots & b_1y_1^q \\ 1 & y_2 & \cdots & b_2y_2^q \\ 1 & z_1 & \cdots & c_1z_1^q \\ \vdots & \vdots & & \vdots \\ 1 & z_m & \cdots & c_mz_m^q \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1+a_1x_1^{p+1} & x_1+a_1x_1^p & \cdots & x_1^{p+1}+a_1 \\ 1+a_2x_2^{p+1} & x_2+a_2x_2^p & \cdots & x_2^{p+1}+a_2 \\ 1+b_1y_1^{p+1} & y_1+b_1y_1^p & \cdots & y_1^{p+1}+b_1 \\ 1+b_2y_2^{p+1} & y_2+b_2y_2^p & \cdots & y_2^{p+1}+b_2 \\ 1+c_1z_1^{p+1} & z_1+c_1z_1^p & \cdots & z_1^{p+1}+c_1 \\ \vdots & \vdots & & \vdots \\ 1+c_pz_p^{p+1} & z_p+c_pz_p^p & \cdots & z_p^{p+1}+c_p \end{pmatrix},$$

where $m = p + q$. □

Proof of Theorem 5.1. We proceed by induction on n. In this proof, we write $f(x_i,x_j)$ (respectively $f(x_i,y_j)$, $f(y_i,y_j)$) instead of $f(x_i,x_j;a_i,a_j)$ (respectively $f(x_i,y_j;a_i,b_j)$, $f(y_i,y_j;b_i,b_j)$).

If $n = 1$, then there is nothing to prove, and, if $n = 2$, then the desired identities are equivalent to the quadratic relations (5.3).

Suppose $n \geqslant 3$. Applying the Desnanot–Jacobi formula (2.4) to the matrix

$$A = \left(\frac{1}{f(x_i,y_j)}\right)_{1\leqslant i,j\leqslant n}.$$

Then, from the induction hypothesis, we have

$$\det A_l^k = \frac{(-1)^{(n-1)(n-2)/2}\prod_{i=3}^n f(x_{k'},x_i)f(y_{l'},y_i)\prod_{3\leqslant i<j\leqslant n} f(x_i,x_j)f(y_i,y_j)}{f(x_{k'},y_{l'})\prod_{i=3}^n f(x_{k'},y_i)f(x_i,y_{l'})\prod_{i,j=3}^n f(x_i,y_j)},$$

where $1 \leqslant k,l \leqslant 2$ and the indices k' and l' are determined by the condition $\{k,k'\} = \{l,l'\} = \{1,2\}$, and

$$\det A_{1,2}^{1,2} = \frac{(-1)^{(n-2)(n-3)/2}\prod_{3\leqslant i<j\leqslant n} f(x_i,x_j)\prod_{3\leqslant i<j\leqslant n} f(y_i,y_j)}{\prod_{i,j=3}^n f(x_i,y_j)}.$$

By canceling the common factors, we see that, in order to prove the identity, it is enough to show

$$-f(x_1,x_2)f(y_1,y_2)=f(x_1,y_2)f(x_2,y_1)-f(x_1,y_1)f(x_2,y_2).$$

This is equivalent to (5.3). □

6. A hyperpfaffian expression

H. Tagawa finds that $\det V^{n,n}(\boldsymbol{x};\boldsymbol{a})$ is expressed by a hyperpfaffian. The aim of this section is to prove this expression.

First we recall the definition of hyperpfaffians (see [11]). Let n and r be positive integers. Define a subset $\mathcal{E}_{rn,n}$ of the symmetric groups $\mathcal{S}_{rn}$ by

$$\mathcal{E}_{rn,n}=\bigl\{\sigma\in\mathcal{S}_{rn}\colon\ \sigma\bigl(n(i-1)+1\bigr)<\sigma\bigl(n(i-1)+2\bigr)<\cdots<\sigma(ni)\ \text{for}\ 1\leqslant i\leqslant r\bigr\}.$$

For example, if $n=r=2$, then $\mathcal{E}_{4,2}$ is composed of the following 6 elements:

$$\mathcal{E}_{4,2}=\bigl\{(1,2,3,4),(1,3,2,4),(1,4,2,3),(3,4,1,2),(2,4,1,3),(2,3,1,4)\bigr\}.$$

Let $a=(a_{i_1,\dots,i_n})_{1\leqslant i_1<\cdots<i_n\leqslant nr}$ be an alternating tensor, i.e. $a_{i_{\sigma(1)},\dots,i_{\sigma(n)}}=\operatorname{sgn}(\sigma)a_{i_1,\dots,i_n}$ for any permutations $\sigma\in\mathcal{S}_{nr}$. The hyperpfaffian of a is, by definition,

$$\operatorname{Pf}^{[n]}(a)=\frac{1}{r!}\sum_{\sigma\in\mathcal{E}_{nr,n}}\operatorname{sgn}(\sigma)\prod_{i=1}^{r}a_{\sigma(n(i-1)+1),\dots,\sigma(ni)}.$$

An alternating 2-tensor a is a skew-symmetric matrix and the hyperpfaffian $\operatorname{Pf}^{[2]}(a)$ is the usual Pfaffian of the skew-symmetric matrix. J.-G. Luque and J.-Y. Thibon [11] computed the following composition of hyperpfaffians by using the Grassmann algebra.

Proposition 6.1. *([11]) Let n and r be positive integers and assume $n=2m$ is even. Given a skew-symmetric matrix $A=(a_{ij})_{1\leqslant i,j\leqslant nr}$, we define an alternating n-tensor $A^{[n]}$ by putting*

$$\bigl(A^{[n]}\bigr)_{i_1,\dots,i_n}=\operatorname{Pf}(a_{i_k,i_l})_{1\leqslant k,l\leqslant n}\quad\textit{for}\ 1\leqslant i_1<\cdots<i_n\leqslant nr.$$

Then we have

$$\operatorname{Pf}^{[n]}\bigl(A^{[n]}\bigr)=\frac{(mr)!}{(m!)^r r!}\operatorname{Pf}(A). \tag{6.1}$$

The main theorem in this section is the following.

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Theorem 6.2. *If n is even, then*

$$\det V^{n,n}(\boldsymbol{x};\boldsymbol{a}) = \mathrm{Pf}^{[n]}\left[\left(1+\prod_{s=1}^{n} a_{i_s}\right)\prod_{1\leqslant s<t\leqslant n}(x_{i_t}-x_{i_s})\right]_{1\leqslant i_1<\cdots<i_n\leqslant 2n}, \tag{6.2}$$

$$\det U^{n,n}\begin{pmatrix}\boldsymbol{x} & \boldsymbol{a}\\ \boldsymbol{y} & \boldsymbol{b}\end{pmatrix} = \mathrm{Pf}^{[n]}\left[\left(\prod_{s=1}^{n} a_{i_s}+\prod_{s=1}^{n} b_{i_s}\right)\prod_{1\leqslant s<t\leqslant n}\det\begin{pmatrix} y_{i_s} & x_{i_s}\\ y_{i_t} & x_{i_t}\end{pmatrix}\right]_{1\leqslant i_1<\cdots<i_n\leqslant 2n}. \tag{6.3}$$

To prove this theorem, we need to compute the following special Pfaffian and hyperpfaffian.

Lemma 6.3. *Let n and r be positive integers and assume $n = 2m$ is even. Then we have*

$$\mathrm{Pf}\left(\frac{(x_j^m - x_i^m)^2}{x_j - x_i}\right)_{1\leqslant i,j\leqslant nr} = \begin{cases}\prod_{1\leqslant i<j\leqslant n}(x_j-x_i) & \text{if } r=1,\\ 0 & \text{if } r\geqslant 2,\end{cases} \tag{6.4}$$

$$\mathrm{Pf}^{[n]}\left[\prod_{1\leqslant s<t\leqslant n}(x_{i_t}-x_{i_s})\right]_{1\leqslant i_1<\cdots<i_n\leqslant nr} = \begin{cases}\prod_{1\leqslant i<j\leqslant n}(x_j-x_i) & \text{if } r=1,\\ 0 & \text{if } r\geqslant 2.\end{cases} \tag{6.5}$$

Proof. There are several ways to prove the identity (6.4). Here we appeal to Theorem 1.1 (1.4). If we take $p=q=r=s=0$ and put $a_i = b_i = x_i^m$ $(1\leqslant i\leqslant nr)$ in (1.4), we have

$$\mathrm{Pf}\left(\frac{(x_j^m - x_i^m)^2}{x_j - x_i}\right)_{1\leqslant i,j\leqslant nr} = \frac{1}{\prod_{1\leqslant i<j\leqslant nr}(x_j-x_i)}\det V^{mr,mr}\bigl(\boldsymbol{x};\boldsymbol{x}^m\bigr)^2.$$

If $r=1$, then $\det V^{m,m}(\boldsymbol{x};\boldsymbol{x}^m)$ is the usual Vandermonde determinant and

$$\det V^{m,m}\bigl(\boldsymbol{x};\boldsymbol{x}^m\bigr) = \prod_{1\leqslant i<j\leqslant n}(x_j-x_i).$$

If $r\geqslant 2$, then the $(m+1)$st column of $V^{mr,mr}(\boldsymbol{x};\boldsymbol{x}^m)$ is the same as the $(rm+1)$st column, so we have $\det V^{mr,mr}(\boldsymbol{x};\boldsymbol{x}^m)=0$. Hence we obtain (6.4).

Next we prove (6.5). Apply Proposition 6.1 to the matrix $A = ((x_j^m - x_i^m)^2/(x_j - x_i))_{1\leqslant i,j\leqslant nr}$. Then it follows from (6.4) that

$$\begin{aligned}\mathrm{Pf}^{[n]}\left[\prod_{1\leqslant s<t\leqslant n}(x_{i_t}-x_{i_s})\right]_{1\leqslant i_1<\cdots<i_n\leqslant nr} &= \mathrm{Pf}^{[n]}\left[\mathrm{Pf}\left(\frac{(x_{i_l}^m - x_{i_k}^m)^2}{x_{i_l}-x_{i_k}}\right)_{1\leqslant k,l\leqslant n}\right]_{1\leqslant i_1<\cdots<i_n\leqslant nr}\\ &= \frac{(mr)!}{(m!)^r r!}\,\mathrm{Pf}\left(\frac{(x_j^m - x_i^m)^2}{x_j - x_i}\right)_{1\leqslant i,j\leqslant nr}.\end{aligned}$$

Again using (6.4), we obtain the desired identity. □

Now we are in position to prove Theorem 6.2.

Proof of Theorem 6.2. The identity (6.3) immediately follows from (6.2) by noting the relation (3.2), so we prove (6.2).

Let $\binom{[2n]}{n}$ denote the set of all n-element subsets of $[2n] = \{1, 2, \dots, 2n\}$. For a subset $I \in \binom{[2n]}{n}$, we put

$$a_I = \prod_{i \in I} a_i, \qquad \Delta(\boldsymbol{x}_I) = \prod_{\substack{i,j \in I \\ i<j}} (x_j - x_i).$$

By the Laplace expansion formula and Vandermonde determinant formula, we have

$$V^{n,n}(\boldsymbol{x}; \boldsymbol{a}) = \sum_{I \in \binom{[2n]}{n}} (-1)^{|I| + \binom{n+1}{2}} a_I \Delta(\boldsymbol{x}_I) \Delta(\boldsymbol{x}_{I^c}),$$

where we write $|I| = \sum_{i \in I} i$ and denote by I^c the complementary subset I in $[2n]$.

On the other hand, by the definition of hyperpfaffians, we see that

$$\begin{aligned} \mathrm{Pf}^{[n]}\big[(1 + a_I)\Delta(\boldsymbol{x}_I)\big]_I &= \frac{1}{2!} \sum_{I \in \binom{[2n]}{n}} (-1)^{|I| + \binom{n+1}{2}} (1 + a_I)\Delta(\boldsymbol{x}_I)(1 + a_{I^c})\Delta(\boldsymbol{x}_{I^c}) \\ &= \left(1 + \prod_{i=1}^{2n} a_i\right) \mathrm{Pf}^{[n]}\Bigg[\prod_{1 \leqslant s < t \leqslant n} (x_{i_t} - x_{i_s})\Bigg]_{1 \leqslant i_1 < \cdots < i_n \leqslant 2n} \\ &\quad + \sum_{I \in \binom{[2n]}{n}} (-1)^{|I| + \binom{n+1}{2}} a_I \Delta(\boldsymbol{x}_I)\Delta(\boldsymbol{x}_{I^c}). \end{aligned}$$

By (6.5), the first term vanishes, and we obtain the desired formula. □

At the end of this section, we should remark that H. Tagawa has a similar hyperpfaffian expression for the case that $p = q$ is odd. It has slightly different from the case $p = q$ is even, but we don't have any general formula when $p \neq q$.

7. Application to Littlewood–Richardson coefficients

In this section, we use the Pfaffian identity (1.5) in Theorem 1.1 and the minor-summation formula [5] to derive a relation between Littlewood–Richardson coefficients.

For three partitions λ, μ and ν, we denote by $\mathrm{LR}^{\lambda}_{\mu,\nu}$ the Littlewood–Richardson coefficient. These numbers $\mathrm{LR}^{\lambda}_{\mu,\nu}$ appear in the following expansions (see [12]):

$$s_\mu(X)s_\nu(X) = \sum_\lambda \mathrm{LR}^\lambda_{\mu,\nu}\, s_\lambda(X),$$

$$s_{\lambda/\mu}(X) = \sum_\nu \mathrm{LR}^\lambda_{\mu,\nu}\, s_\nu(X),$$

$$s_\lambda(X,Y) = \sum_{\mu,\nu} \mathrm{LR}^\lambda_{\mu,\nu}\, s_\mu(X)s_\nu(Y).$$

We are concerned with the Littlewood–Richardson coefficients involving rectangular partitions. Let $\square(a,b)$ denote the partition whose Young diagram is the rectangle $a \times b$, i.e.

$$\square(a,b) = (b^a) = (\underbrace{b,\dots,b}_{a}).$$

For a partition $\lambda \subset \square(a,b)$, we define a partition $\lambda^\dagger = \lambda^\dagger(a,b)$ by

$$\lambda^\dagger_i = b - \lambda_{a+1-i} \quad (1 \leqslant i \leqslant a).$$

This partition $\lambda^\dagger$ is the complement of λ in the rectangle $\square(a,b)$.

Okada [13] used the special case of the identities (1.3) and (1.4) (i.e., the case of $p = q = 0$ and $p = q = r = s = 0$) to prove the following proposition.

Proposition 7.1. *Let n be a positive integer and let e and f be nonnegative integers.*

(1) *For partitions μ, ν, we have*

$$\mathrm{LR}^{\square(n,e)}_{\mu,\nu} = \begin{cases} 1 & \text{if } \nu = \mu^\dagger(n,e), \\ 0 & \text{otherwise.} \end{cases} \tag{7.1}$$

(2) *For a partition λ of length $\leqslant 2n$, we have*

$$\mathrm{LR}^\lambda_{\square(n,e),\square(n,f)} = \begin{cases} 1 & \text{if } \lambda_{n+1} \leqslant \min(e,f) \text{ and } \lambda_i + \lambda_{2n+1-i} = e+f \quad (1 \leqslant i \leqslant n), \\ 0 & \text{otherwise.} \end{cases} \tag{7.2}$$

The main result of this section is the following theorem, which generalizes (7.2).

Theorem 7.2. *Let n be a positive integer and let e and f be nonnegative integers. Let λ and μ be partitions such that the length $l(\lambda) \leqslant 2n$ and $\mu \subset \square(n,e)$. Then we have*

(1) $\mathrm{LR}^\lambda_{\mu,\square(n,f)} = 0$ *unless*

$$\lambda_n \geqslant f \quad \text{and} \quad \lambda_{n+1} \leqslant \min(e,f). \tag{7.3}$$

(2) *If λ satisfies the above condition* (7.3) *and we define two partitions α and β by*

$$\alpha_i = \lambda_i - f, \qquad \beta_i = e - \lambda_{2n+1-i} \quad (1 \leqslant i \leqslant n), \tag{7.4}$$

then we have

$$\mathrm{LR}^{\lambda}_{\mu,\square(n,f)} = \mathrm{LR}^{\beta}_{\alpha,\mu^{\dagger}(n,e)}.$$

In particular, $\mathrm{LR}^{\lambda}_{\mu,\square(n,f)} = 0$ *unless* $\alpha \subset \beta$.

In particular, if $\mu = \square(n,e)$ is a rectangle, then this theorem reduces to (7.2), because $\mathrm{LR}^{\alpha}_{\beta,\emptyset} = \delta_{\alpha,\beta}$. If μ is a near-rectangle, then we have the following corollary by using Pieri's rule [12, (5.16), (5.17)].

Corollary 7.3. *Suppose that a partition* $\lambda \subset \square(2n, e+f)$ *satisfies the condition* (7.3) *in Theorem* 7.2. *Define two partitions* α *and* β *by* (7.4). *Then we have*

$$\mathrm{LR}^{\lambda}_{(e^{n-1},e-k),(f^n)} = \begin{cases} 1 & \text{if } \beta/\alpha \text{ is a horizontal strip of length } k, \\ 0 & \text{otherwise}; \end{cases}$$

$$\mathrm{LR}^{\lambda}_{(e^{n-k},(e-1)^k),(f^n)} = \begin{cases} 1 & \text{if } \beta/\alpha \text{ is a vertical strip of length } k, \\ 0 & \text{otherwise.} \end{cases}$$

In order to prove Theorem 7.2, we substitute

$$a_i = x_i^{e+p+n}, \qquad b_i = x_i^{f+r+n}, \qquad c_i = z_i^{e+p+n}, \qquad d_i = w_i^{f+r+n} \tag{7.5}$$

in the Pfaffian identity (1.4). By the bi-determinant definition of Schur functions, we have

$$\det V^{p,q}\big(\boldsymbol{x}; \boldsymbol{x}^k\big) = \begin{cases} s_{\square(q,k-p)}(\boldsymbol{x})\Delta(\boldsymbol{x}) & \text{if } k \geqslant p, \\ 0 & \text{if } k < p, \end{cases}$$

where $\Delta(\boldsymbol{x}) = \prod_{1 \leqslant i < j \leqslant n}(x_j - x_i)$. Hence, under the substitution (7.5), the identity (1.4) gives us the following Pfaffian identity.

Proposition 7.4. *We have*

$$\begin{aligned} &\frac{1}{\Delta(\boldsymbol{x})} \operatorname{Pf}\big((x_j - x_i) s_{\square(q+1,e+n-1)}(x_i, x_j, \boldsymbol{z}) s_{\square(s+1,f+n-1)}(x_i, x_j, \boldsymbol{w})\big)_{1 \leqslant i,j \leqslant 2n} \\ &\quad = s_{\square(q,e+n)}(\boldsymbol{z})^{n-1} s_{\square(s,f+n)}(\boldsymbol{w})^{n-1} s_{\square(n+q,e)}(\boldsymbol{x}, \boldsymbol{z}) s_{\square(n+s,f)}(\boldsymbol{x}, \boldsymbol{w}). \end{aligned} \tag{7.6}$$

Remark 7.5. If we substitute

$$a_i = x_i^{e+p+n}, \qquad b_i = y_i^{e+p+n} \quad (1 \leqslant i \leqslant n)$$

in the determinant identity (1.3), then we have

$$\begin{aligned} &\frac{1}{\Delta(\boldsymbol{x})\Delta(\boldsymbol{y})} \det\big(s_{\square(q+1,e+n-1)}(x_i, y_j, \boldsymbol{z})\big)_{1 \leqslant i,j \leqslant n} \\ &\quad = (-1)^{n(n-1)/2} s_{\square(q,e+n)}(\boldsymbol{z})^{n-1} s_{\square(q+n,e)}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}). \end{aligned} \tag{7.7}$$

The special case ($q = e + n - 1$) of this identity is given in [10, Proposition 8.4.3], and the proof there works in the general case.

If we take $q = s = 0$ in (7.6), we have

$$\begin{aligned} &\frac{1}{\Delta(\boldsymbol{x})} \operatorname{Pf}\bigl((x_j - x_i) h_{e+n-1}(x_i, x_j, \boldsymbol{z}) h_{f+n-1}(x_i, x_j, \boldsymbol{w})\bigr)_{1 \leqslant i, j \leqslant 2n} \\ &\quad = s_{\square(n,e)}(\boldsymbol{x}, \boldsymbol{z}) s_{\square(n,f)}(\boldsymbol{x}, \boldsymbol{w}). \end{aligned} \tag{7.8}$$

We use the minor-summation formula [5] to expand the left hand side in the Schur function bases $\{s_\lambda(\boldsymbol{x})\}$.

Lemma 7.6. *Let $b_{k,l}$ be the coefficient of $x^k y^l$ in*

$$(y - x) h_{e+n-1}(x, y, \boldsymbol{z}) h_{f+n-1}(x, y, \boldsymbol{w}).$$

Then we have $b_{kl} = -b_{lk}$, and b_{kl}, $k < l$, is given by

$$b_{kl} = \sum_{i,j} h_i(\boldsymbol{z}) h_j(\boldsymbol{w}),$$

where the sum is taken over all pairs of integers (i, j) satisfying

$$\begin{aligned} &i + j = (e + n - 1) + (f + n - 1) + 1 - k - l, \\ &0 \leqslant i \leqslant (e + n - 1) - k, \quad 0 \leqslant j \leqslant (f + n - 1) - k. \end{aligned}$$

Note that $b_{kl} = 0$ unless $0 \leqslant k, l \leqslant e + f + 2n - 1$.

Proof. By using the relation

$$h_r(x, y, \boldsymbol{z}) = \sum_{a,b \geqslant 0} x^a y^b h_{r-a-b}(\boldsymbol{z}),$$

we see that

$$b_{kl} = \Biggl(\sum_{\substack{0 \leqslant a,b \leqslant e+n-1 \\ 0 \leqslant c,d \leqslant f+n-1 \\ a+c=k,\ b+d=l-1}} - \sum_{\substack{0 \leqslant a,b \leqslant e+n-1 \\ 0 \leqslant c,d \leqslant f+n-1 \\ a+c=k-1,\ b+d=l}} \Biggr) h_{(e+n-1)-a-b}(\boldsymbol{z}) h_{(f+n-1)-c-d}(\boldsymbol{w}).$$

Let $b_{kl}(i, j)$ be the coefficient of $h_i(\boldsymbol{z}) h_j(\boldsymbol{w})$ in b_{kl}. Then, by considering the homogeneous degree, we see that $b_{kl}(i, j) = 0$ unless $i + j = (e + n - 1) + (f + n - 1) + 1 - k - l$, $0 \leqslant i \leqslant e + n - 1$ and $0 \leqslant j \leqslant f + n - 1$.

Now we assume

$$i+j=(e+n-1)+(f+n-1)+1-k-l,$$
$$0\leqslant i\leqslant e+n-1,\quad 0\leqslant j\leqslant f+n-1.$$

If we put

$$S_1=\left\{\begin{pmatrix}a\\b\\c\\d\end{pmatrix}\in\mathbb{N}^4\colon\ \begin{matrix}a+c=k,\\ b+d=l-1,\\ a+b=(e+n-1)-i,\\ c+d=(f+n-1)-j\end{matrix}\right\},$$
$$S_2=\left\{\begin{pmatrix}a'\\b'\\c'\\d'\end{pmatrix}\in\mathbb{N}^4\colon\ \begin{matrix}a'+c'=k-1,\\ b'+d'=l,\\ a'+b'=(e+n-1)-i,\\ c'+d'=(f+n-1)-j\end{matrix}\right\},$$

where $\mathbb{N}$ denotes the set of nonnegative integers, then we have

$$b_{kl}(i,j)=\#S_1-\#S_2.$$

The solutions to the equations in S_1 and S_2 are given by

$$\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}t+(e+n-1)-i-l+1\\ -t+l-1\\ -t+k+l-1-(e+n-1)+i\\ t\end{pmatrix},$$
$$\begin{pmatrix}a'\\b'\\c'\\d'\end{pmatrix}=\begin{pmatrix}t+(e+n-1)-i-l\\ -t+l\\ -t+k+l-1-(e+n-1)+i\\ t\end{pmatrix}.$$

Hence we see that

$$\#S_1=\#\{t\in\mathbb{Z}\colon\ t\geqslant a_0,\ t\leqslant b_0,\ t\leqslant c_0,\ t\geqslant d_0\},$$
$$\#S_2=\#\{t\in\mathbb{Z}\colon\ t\geqslant a_0',\ t\leqslant b_0',\ t\leqslant c_0',\ t\geqslant d_0'\},$$

where

$$\begin{pmatrix}a_0\\b_0\\c_0\\d_0\end{pmatrix}=\begin{pmatrix}l-1+i-(e+n-1)\\ l-1\\ k+l-1-(e+n-1)+i\\ 0\end{pmatrix},$$
$$\begin{pmatrix}a_0'\\b_0'\\c_0'\\d_0'\end{pmatrix}=\begin{pmatrix}l+i-(e+n-1)\\ l\\ k+l-1-(e+n-1)+i\\ 0\end{pmatrix}.$$

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We compute $\#S_1$ and $\#S_2$ in the following four cases:

(a) $i \leqslant (e+n-1)-k$ and $j \leqslant (f+n-1)-k$.
(b) $i \leqslant (e+n-1)-k$ and $j > (f+n-1)-k$.
(c) $i > (e+n-1)-k$ and $j \leqslant (f+n-1)-k$.
(d) $i > (e+n-1)-k$ and $j > (f+n-1)-k$.

Here we note that

$$j \leqslant (f+n-1)-k \quad \text{if and only if} \quad l+i-(e+n-1)-1 \geqslant 0,$$

and that

$$a_0-d_0 = l+i-(e+n-1)-1, \qquad b_0-c_0 = (e+n-1)-i-k,$$
$$a'_0-d'_0 = l+i-(e+n-1) = a_0-d_0+1,$$
$$b'_0-c'_0 = (e+n-1)-i-k+1 = b_0-c_0+1.$$

Hence we see that, if $i \leqslant (e+n-1)-k$, then $b_0 \geqslant c_0$ and $b'_0 > c'_0$, and that, if $i \leqslant l-k$, then $a_0 \geqslant d_0$ and $a'_0 > d'_0$.

In Case (a), we have $a_0 \geqslant d_0$, $b_0 \geqslant c_0$, $a'_0 > d'_0$ and $b'_0 > c'_0$, so

$$\#S_1 = \#\{t \in \mathbb{Z}\colon\ t \geqslant a_0,\ t \leqslant c_0\} = c_0-a_0+1 = k+1,$$
$$\#S_2 = \#\{t \in \mathbb{Z}\colon\ t \geqslant a'_0,\ t \leqslant c'_0\} = c'_0-a'_0+1 = k.$$

Hence we have $b_{kl}(p,q)=1$. (This argument holds if $k=0$.) In Case (b), we have $b_0 \geqslant c_0$, $b'_0 > c'_0$, $a_0 < d_0$ and $a'_0 \leqslant d'_0$, so

$$\#S_1 = \#\{t \in \mathbb{Z}\colon\ t \geqslant d_0,\ t \leqslant c_0\} = c_0-d_0+1 = k+l+i-(e+n-1),$$
$$\#S_2 = \#\{t \in \mathbb{Z}\colon\ t \geqslant d'_0,\ t \leqslant c'_0\} = c'_0-d'_0+1 = k+l+i-(e+n-1).$$

Hence we have $b_{kl}(i,j)=0$. Similarly, in Case (c), we have $b_{kl}(i,j)=0$. In Case (d), we have $i+j > (e+n-1)+(f+n-1)-k-l+1$, which contradicts to the assumption $i+j = (e+n-1)+(f+n-1)-k-l+1$.

This completes the proof. □

Here we recall the minor summation formula [5].

Lemma 7.7. *Let X be a $2n \times N$ matrix and A be an $N \times N$ skew-symmetric matrix. Then we have*

$$\sum_I \operatorname{Pf} \Delta_I^I(A) \det \Delta_I(X) = \operatorname{Pf}(XA^tX),$$

where I runs over all $2n$-element subsets of $[N]$.

By applying this minor-summation formula, we obtain

Proposition 7.8. *Let $B = (b_{ij})_{i,j\geqslant 0}$ be the skew-symmetric matrix, whose entries b_{ij} are given in Lemma* 7.6. *Then, for a partition λ of length $\leqslant 2n$, we have*

$$\sum_{\substack{\mu\subset\square(n,e)\\ \nu\subset\square(n,f)}} \mathrm{LR}^{\lambda}_{\mu,\nu}\, s_{\mu^{\dagger}(n,e)}(\boldsymbol{z}) s_{\nu^{\dagger}(n,f)}(\boldsymbol{w}) = \mathrm{Pf}\,\Delta^{I(\lambda)}_{I(\lambda)}(B). \tag{7.9}$$

Proof. Apply Lemma 7.7 to the matrix $X = (x_i^k)_{1\leqslant i\leqslant 2n, k\geqslant 0}$ and the skew-symmetric matrix B. Since $\det \Delta_{I(\lambda)}(X)/\Delta(\boldsymbol{x}) = s_{\lambda}(\boldsymbol{x})$, the left hand side of (7.6) becomes

$$\frac{1}{\Delta(\boldsymbol{x})}\mathrm{Pf}\big((x_j - x_i)h_{e+n-1}(x_i, x_j, \boldsymbol{z})h_{f+n-1}(x_i, x_j, \boldsymbol{w})\big)_{1\leqslant i,j\leqslant 2n} = \sum_{\lambda} \mathrm{Pf}\,\Delta^{I(\lambda)}_{I(\lambda)}(B) s_{\lambda}(\boldsymbol{x}),$$

where λ runs over all partitions of length $\leqslant 2n$. Here we note that $\mathrm{Pf}\,\Delta^{I(\lambda)}_{I(\lambda)}(B) = 0$ unless $\lambda \subset \square(2n, e+f)$.

On the other hand, the right hand side of (7.6) is expanded in the Schur function basis $\{s_{\lambda}(\boldsymbol{x})\}$ as follows. It follows from (7.1) that

$$s_{\square(n,e)}(\boldsymbol{x}, \boldsymbol{z}) = \sum_{\mu\subset\square(n,e)} s_{\mu}(\boldsymbol{x}) s_{\mu^{\dagger}(n,e)}(\boldsymbol{z}),$$
$$s_{\square(n,f)}(\boldsymbol{x}, \boldsymbol{w}) = \sum_{\nu\subset\square(n,f)} s_{\nu}(\boldsymbol{x}) s_{\nu^{\dagger}(n,f)}(\boldsymbol{w}).$$

Hence we see that the right hand side of (7.8) becomes

$$\begin{aligned} s_{\square(n,e)}(\boldsymbol{x}, \boldsymbol{z}) s_{\square(n,f)}(\boldsymbol{x}, \boldsymbol{w}) &= \sum_{\substack{\mu\subset\square(n,e)\\ \nu\subset\square(n,f)}} s_{\mu}(\boldsymbol{x}) s_{\nu}(\boldsymbol{x}) s_{\mu^{\dagger}(n,e)}(\boldsymbol{z}) s_{\nu^{\dagger}(n,f)}(\boldsymbol{w}) \\ &= \sum_{\lambda}\Bigg(\sum_{\substack{\mu\subset\square(n,e)\\ \nu\subset\square(n,f)}} \mathrm{LR}^{\lambda}_{\mu,\nu}\, s_{\mu^{\dagger}(n,e)}(\boldsymbol{z}) s_{\nu^{\dagger}(n,f)}(\boldsymbol{w})\Bigg) s_{\lambda}(\boldsymbol{x}). \end{aligned}$$

Comparing the coefficient of $s_{\lambda}(\boldsymbol{x})$ on both sides of (7.8) completes the proof of (7.9). $\square$

Now we can finish the proof of Theorem 7.2.

Proof of Theorem 7.2. In the above argument, we take $p \geqslant n$ and $r = 0$. In this case, the variables $\boldsymbol{w}$ disappear and we see that

$$b_{kl} = \begin{cases} h_{(e+n-1)+(f+n-1)+1-k-l}(\boldsymbol{z}) & \text{if } 0\leqslant k\leqslant \min(e+n-1, f+n-1) \text{ and} \\ & \quad l \geqslant f+n-1, \\ 0 & \text{otherwise} \end{cases}$$

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and the equation (7.9) becomes

$$\sum_{\mu\subset\Box(n,e)} \mathrm{LR}^{\lambda}_{\mu,\Box(n,f)}\, s_{\mu^{\dagger}(n,e)}(z) = \mathrm{Pf}\,\Delta^{I(\lambda)}_{I(\lambda)}(B).$$

The skew-symmetric matrix B has the form

$$B = \begin{pmatrix} O & C & O \\ -{}^tC & O & O \\ O & O & O \end{pmatrix}, \qquad C = \big(h_{e+n-1-i-j}(z)\big)_{0\leqslant i\leqslant f+n-1,\,0\leqslant j\leqslant e+n-1}.$$

From the relation (3.5), we see that the sub-Pfaffian $\mathrm{Pf}\,\Delta^{I(\lambda)}_{I(\lambda)}(B)$ vanishes unless

$$\lambda_{n+1} + n - 1 \leqslant \min(e+n-1, f+n-1), \qquad \lambda_n + n \geqslant f + n,$$

i.e.,

$$\lambda_{n+1} \leqslant \min(e, f), \qquad \lambda_n \geqslant f.$$

If these conditions are satisfied, then we have

$$\begin{aligned} \mathrm{Pf}\,\Delta^{I(\lambda)}_{I(\lambda)}(B) &= (-1)^{n(n-1)/2} \det\big(h_{\beta_i-\alpha_{n+1-j}-i+(n+1-j)}(z)\big)_{1\leqslant i,j\leqslant n} \\ &= (-1)^{n(n-1)/2}(-1)^{n(n-1)/2} \det\big(h_{\beta_i-\alpha_j-i+j}(z)\big)_{1\leqslant i,j\leqslant n} \\ &= s_{\beta/\alpha}(z). \end{aligned}$$

Hence we have

$$\sum_{\mu\subset\Box(n,e)} \mathrm{LR}^{\lambda}_{\mu,\Box(n,f)}\, s_{\mu^{\dagger}(n,e)}(z) = s_{\beta/\alpha}(z).$$

Comparing the coefficients of $s_{\mu^{\dagger}(n,e)}(z)$ completes the proof. □

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