Refined Enumerations of Totally Symmetric Self-Complementary Plane Partitions and Lattice Path Combinatorics

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Abstract

This article is a short explanation of some of the results obtained in my papers "On refined enumerations of totally symmetric self-complementary plane partitions I, II". We give Pfaffian expressions for some of the conjectures in the paper "Self-complementary totally symmetric plane partitions" (*J. Combin. Theory Ser. A* **42**, 277–292) by Mills, Robbins and Rumsey, using the lattice path method.

1 Introduction

In the paper [8] Mills, Robbins and Rumsey presented several conjectures on the enumeration of the totally symmetric self-complementary plane partitions. The aim of this article is to obtain a Pfaffian expressions for the refined enumeration and doubly refined enumeration of the totally symmetric self-complementary plane partitions (see Theorem 1.4). In [4, 5], we obtain more Pfaffian or determinant expressions, and certain constant term identities for the conjectures.

A *plane partition* is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers such that π has finite support (i.e. finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum_{i,j\geq 1}\pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n, or π has *weight* n. A *part* of a plane partition $\pi = (\pi_{ij})_{i,j\geq 1}$ is a positive entry $\pi_{ij} > 0$. The *shape* of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row. Consider the elements of $\mathbb{P}^3$, regarded as the lattice points of $\mathbb{R}^3$ in the positive orthant. The *Ferrers graph* $F(\pi)$ of π is the set of all lattice points $(i,j,k) \in \mathbb{P}^3$ such that $k \leq \pi_{ij}$. A subset F of $\mathbb{P}^3$ is a Ferrers graph if and only if it satisfies

$$x_1 \leq x_2,\ y_1 \leq y_2,\ z_1 \leq z_2 \text{ and } (x_2,y_2,z_2) \in F \Rightarrow (x_1,y_1,z_1) \in F.$$

Hereafter we identify a plane partition and its Ferrers graph, and write π for $F(\pi)$. The symmetric group S_3 is acting on $\mathbb{P}^3$ as permutations of the coordinate axes. A plane partition is said to be *totally symmetric* if its Ferrers graph is mapped to itself under all 6 permutations in S_3.

A plane partition $\pi \subseteq X_{r,s,t} := [r] \times [s] \times [t]$ is (r,s,t)-self-complementary if we have, for all $p \in X_{r,s,t}$, $p \in \pi$ if and only if $\sigma_{r,s,t}(p) \notin \pi$. Let $\mathscr{T}_n$ denote the set of all plane partitions which is contained in the cube X_{2n}, $(2n,2n,2n)$-self-complementary and totally symmetric.

In [8] Mills, Robbins and Rumsey have introduced a class $\mathscr{B}_n$ of triangular shifted plane partitions

$$\begin{matrix} b_{11} & b_{12} & \dots & b_{1,n-1} \\ & b_{22} & \dots & b_{2,n-1} \\ & & \ddots & \vdots \\ & & & b_{n-1,n-1} \end{matrix}$$

whose parts are $\leq n$, weakly decreasing along rows and columns, and all parts in row i are $\geq n-i$. For example, $\mathscr{B}_3$ consists of the following seven elements.

$$\begin{matrix}3 & 3\\ & 3\end{matrix}\quad\begin{matrix}3 & 3\\ & 2\end{matrix}\quad\begin{matrix}3 & 3\\ & 1\end{matrix}\quad\begin{matrix}3 & 2\\ & 2\end{matrix}\quad\begin{matrix}3 & 2\\ & 1\end{matrix}\quad\begin{matrix}2 & 2\\ & 2\end{matrix}\quad\begin{matrix}2 & 2\\ & 1\end{matrix}$$

They have established an bijection between $\mathscr{T}_n$ and $\mathscr{B}_n$.

Let μ be a strict partition. A *shifted plane partition* τ of *shifted shape* μ is an arbitrary filling of the cells of μ with nonnegative integers such that each entry is weakly decreasing in rows and columns. In this article we allow parts to be zero for shifted plane partitions of a fixed shifted shape μ. Here we consider a more general set $\mathscr{B}_{n,m}$ of shifted plane partitions which appeared in [7, Theorem 1].

Definition 1.1. Let m and $n \geq 1$ be nonnegative integers. Let $\mathscr{B}_{n,m}$ denote the set of shifted plane partitions $b = (b_{ij})_{1\leq i\leq j}$ subject to the constraints that

(B1) the shifted shape of b is $(n+m-1, n+m-2, \ldots, 2, 1)$;

(B2) $\max\{n-i, 0\} \leq b_{ij} \leq n$ for $1 \leq i \leq j \leq n+m-1$.

The main object we study in this article is the following set $\mathscr{P}_{n,m}$, which is bijective with the set $\mathscr{B}_{n,m}$ defined above.

Definition 1.2. Let m and $n \geq 1$ be nonnegative integers. Let $\mathscr{P}_{n,m}$ denote the set of plane partitions $c = (c_{ij})_{1\leq i,j}$ subject to the constraints that

(C1) c has at most n columns;

(C2) c is column-strict and each part in the jth column does not exceed $n+m-j$.

If a part in the jth column of c is equal to $n+m-j$ (that can happen only in the first row, i.e. $c_{1j} = n+m-j$), we call the part a *saturated part*.

The important fact is that we construct a bijection between $\mathscr{B}_{n,m}$ and $\mathscr{P}_{n,m}$ in [4]. By this bijection, the statistics on $\mathscr{B}_{n,m}$ defined by Mills, Robbins and Rumsey in [8] correspond to the following statistics $\overline{U}_r(c)$ on $\mathscr{P}_{n,m}$.

Definition 1.3. For $c \in \mathscr{P}_{n,m}$, let $\overline{U}_r(c)$ be the number of parts equal to r plus the number of saturated parts less than r, i.e.

$$\overline{U}_r(c) = \sharp\{(i,j) : c_{ij} = r\} + \sharp\{1 \leq k < r : c_{1,n+m-k} = k\}. \tag{1.1}$$

Especially $\overline{U}_1(c)$ is the number of 1's in c and $\overline{U}_{n+m}(c)$ is the number of saturated parts in c. It is also easy to see that $\overline{U}_{n+m-1}(c) = \overline{U}_{n+m}(c)$ since, if a part of $c \in \mathscr{P}_{n,m}$ is equal to $n+m-1$, then it is saturated.

Let $\bar{S}_n = (\bar{s}_{ij})_{1\leq i,j\leq n}$ be the skew-symmetric matrix of size n whose (i,j)th entry $\bar{s}_{ij}$ is $(-1)^{j-i-1}$ for $1 \leq i < j \leq n$. Let $B^N_{n,m}(t,u) = (b^{(m)}_{ij}(t,u))_{0\leq i\leq n-1,\ 0\leq j\leq n+N-1}$ be the $n \times (n+N)$ matrix whose (i,j)th entry is

$$b^{(m)}_{ij}(t,u) = \begin{cases} \delta_{0,j} & \text{if } i+m=0,\\ \binom{i+m-1}{j-i} + \binom{i+m-1}{j-i-1}tu & \text{if } i+m=1,\\ \binom{i+m-2}{j-i} + \binom{i+m-2}{j-i-1}(t+u) + \binom{i+m-2}{j-i-2}tu & \text{otherwise.}\end{cases} \tag{1.2}$$

For example,

$$B^2_{3,0}(t,u) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & tu & 0 & 0\\ 0 & 0 & 1 & t+u & tu \end{pmatrix}.$$

We define the $n \times (n+N)$ matrices $B^N_{n,m}(t) = B^N_{n,m}(t,1)$ and $B^N_{n,m} = B^N_{n,m}(1)$. The results of this article is the following theorem.

Theorem 1.4. Let m and $n \geq 1$ be non-negative integers, and let N be an even integer such that $N \geq n+m-1$.

(i) If r is a positive integer such that $2 \leq r \leq n+m$, then the generating function for all plane partitions $c \in \mathscr{P}_{n,m}$ with the weight $t^{\overline{U}_1(c)}u^{\overline{U}_r(c)}$ is

$$\sum_{c \in \mathscr{P}_{n,m}} t^{\overline{U}_1(c)}u^{\overline{U}_r(c)} = \operatorname{Pf}\begin{pmatrix} O_n & J_n B^N_{n,m}(t,u) \\ -{}^tB^N_{n,m}(t,u)J_n & \bar{S}_{n+N} \end{pmatrix}. \tag{1.3}$$

(ii) If r is a positive integer such that $1 \leq r \leq n+m$, then the generating function for all plane partitions $c \in \mathscr{P}_{n,m}$ with the weight $t^{\overline{U}_r(c)}$ is given by

$$\sum_{c \in \mathscr{P}_{n,m}} t^{\overline{U}_r(c)} = \operatorname{Pf}\begin{pmatrix} O_n & J_n B^N_{n,m}(t) \\ -{}^tB^N_{n,m}(t)J_n & \bar{S}_{n+N} \end{pmatrix}. \tag{1.4}$$

Now we assign weight

$$\boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c = \prod_{k=1}^{m+n} t_k^{\overline{U}_k(c)} \prod_{i \geq 1} x_i^{\sharp i\text{'s in } c}$$

to each $c \in \mathscr{P}_{n,m}$. We prove Theorem 1.4 from the minor summation formula [6] and the following thorem, which can be proved with the lattice path method.

Theorem 1.5. Let m and $n \geq 1$ be non-negative integers, and put $N = n+m$. Let λ be a partition with $\ell(\lambda) \leq n$. Then the generating function of all plane partitions $c \in \mathscr{P}_{n,m}$ of shape λ' with the weight $\boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c$ is given by

$$\sum_{\substack{c \in \mathscr{P}_{n,m} \\ \operatorname{sh}(c)=\lambda'}} \boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c = \det\left(e^{(N-i)}_{\lambda_j-j+i}(t_1x_1,\dots,t_{N-i-1}x_{N-i-1},T_{N-i}x_{N-i})\right)_{1 \leq i,j \leq n}, \tag{1.5}$$

where $T_i = \prod_{k=i}^N t_k$.

In fact, we give a lattice path realization of each $c \in \mathscr{P}_{n,m}$. Let $V = \{(x,y) \in \mathbb{N}^2 : 0 \leq y \leq x\}$ be the vertex set, and direct an edge from u to v whenever $v-u=(1,-1)$ or $(0,-1)$.

(i) We assign the weight

$$\begin{cases} \prod_{k=j}^N t_k \cdot x_j & \text{if } j = i, \\ t_jx_j & \text{if } j < i, \end{cases}$$

to the horizontal edge from $u=(i,j)$ to $v=(i+1,j-1)$.

(ii) We assign the weight 1 to the vertical edge from $u=(i,j)$ to $v=(i,j-1)$.

Let $u_j = (N-j, N-j)$ and $v_j = (\lambda_j+N-j, 0)$ for $j = 1,\dots,n$, and let $\boldsymbol{u} = (u_1,\dots,u_n)$ and $\boldsymbol{v} = (v_1,\dots,v_n)$. We claim that the $c \in \mathscr{P}_{n,m}$ of shape λ' can be identified as n-tuples of nonintersecting D-paths in $\mathscr{P}(\boldsymbol{u},\boldsymbol{v})$. For example, the plane partition

8	**8**	**7**	5	**5**	3	**3**
7	7	6	3	3	2	
5	5	5	2	2		
3	2	2	1	1		
2	1	1				
1						

corresponds the lattice paths illustrated in Figure 1.

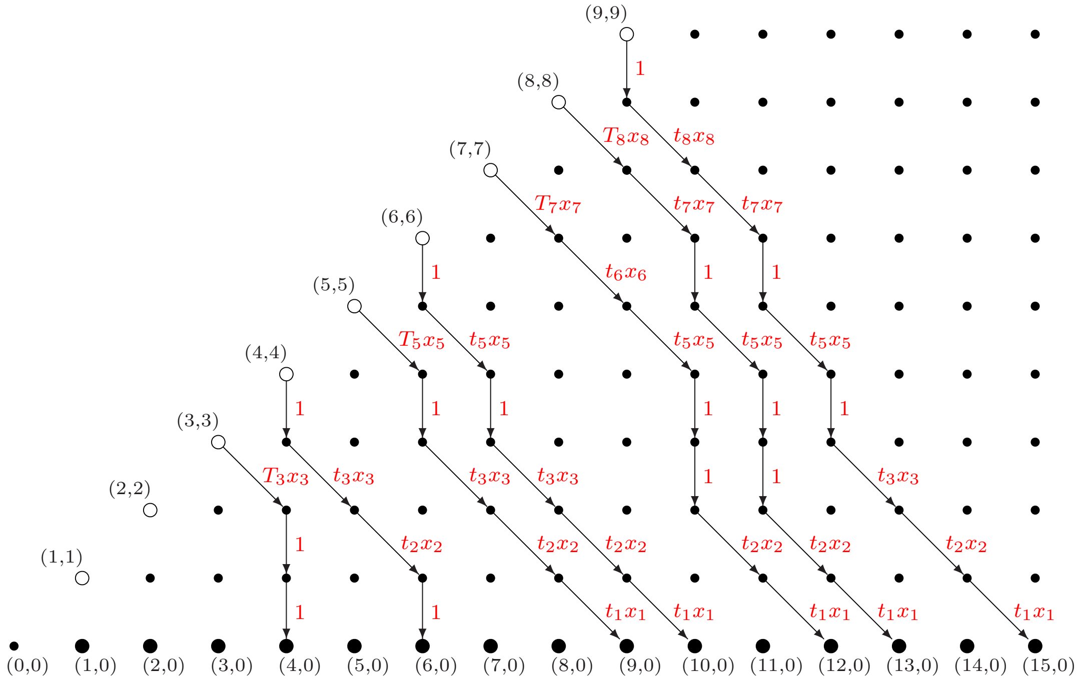

Figure 1: Lattice Paths ($n = 7$, $m = 3$, $\lambda' = (65^2 4^2 21)$, $T_i = \prod_{k=i}^{n+m} t_k$.)

References

[1] G.E. Andrews and W.H. Burge, "Determinant identities", *Pacific J. Math.* **158** (1993), 1–14.

[2] D.M. Bressound, *Proofs and Confirmations*, Cambridge U.P.

[3] I. Gessel and G. Viennot, *Determinants, Paths, and Plane Partitions*, preprint (1989).

[4] M. Ishikawa, "On refined enumerations of totally symmetric self-complementary plane partitions I", `arXiv:math.CO/0602068`.

[5] M. Ishikawa, "On refined enumerations of totally symmetric self-complementary plane partitions II", `arXiv:math.CO/0606082`.

[6] M. Ishikawa and M. Wakayama, "Applications of the minor summation formula III: Plücker relations, lattice pathes and Pfaffians", `arXiv:math.CO/0312358`, *J. Combin. Theory Ser. A* **113** (2006) 113-155.

[7] C. Krattenthaler, "Determinant identities and a generalization of the number of totally symmetric self-complementary plane partitions", *Electron. J. Combin.* **4(1)** (1997), #R27.

[8] W.H. Mills, D.P. Robbins and H. Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser. A* **42**, (1986), 277–292.

[9] R.P. Stanley, "Symmetries of plane partitions", *J. Combin. Theory Ser. A* **43**, (1986), 103–113.

[10] J.R. Stembridge, "Nonintersecting paths, Pfaffians, and plane partitions" *Adv. math.*, **83** (1990), 96–131.

[11] D. Zeilberger, "A constant term identity featuring the ubiquitous (and mysterious) Andrews-Mills-Robbins-Rumsey numbers", *J. Combin. Theory Ser. A* **66** (1994), 17–27.