On refined enumerations of totally symmetric self-complementary plane partitions I

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Abstract

In this paper we give Pfaffian expressions and constant term identities for three conjectures (i.e. Conjecture 2, Conjecture 3 and Conjecture 7) by Mills, Robbins and Rumsey in the paper "Self-complementary totally symmetric plane partitions" *J. Combin. Theory Ser. A* **42**, 277–292) concerning the refined enumeration problems of totally symmetric self-complementary plane partitions. We also present some new conjectures and give Pfaffian expressions and constant term identities for them. But evaluation problem of these Pfaffians are still difficult.

Contents

1   Introduction

In the paper [26] Mills, Robbins and Rumsey presented several conjectures on the enumeration of the totally symmetric self-complementary plane partitions. G.E. Andrews ([2]) settled the conjecture ([26, Conjecture 1]) on the cardinality of the totally

symmetric self-complementary plane partitions of size n (see also [32]). D. Zeilberger gave a constant term identity of this cardinality in [35]. The aim of this paper is to give Pfaffian expressions of the other conjectures in [26] which are the enumeration with some weight or enumeration of some subset. We also generalize Zeilberger's constant term identity, and show that each enumeration correspond to each of the classical Littlwood type identities for Schur functions.

In [26] Mills, Robbins and Rumsey have introduced a class $\mathscr{B}_n$ of triangular shifted plane partitions

$$
\begin{array}{ccccc}
b_{11} & b_{12} & \dots & b_{1,n-1} \\
& b_{22} & \dots & b_{2,n-1} \\
& & \ddots & \vdots \\
& & & b_{n-1,n-1}
\end{array}
$$

whose parts are $\leq n$, weakly decreasing along rows and columns, and all parts in row i are $\geq n - i$. For example, $\mathscr{B}_3$ consists of the following seven elements.

$$
\begin{array}{ccccccc}
3\ 3 & 3\ 3 & 3\ 3 & 3\ 2 & 3\ 2 & 2\ 2 & 2\ 2 \\
3 & 2 & 1 & 2 & 1 & 2 & 1
\end{array}
$$

They have established an bijection between the totally symmetric self-complementary plane partitions of size n and the elements of $\mathscr{B}_n$ (see Section 3), and defined a new statistics $U_r(b)$ for an element $b \in \mathscr{B}_n$ and $r = 1, \dots, n$, i.e., for a $b = (b_{ij})_{1 \leq i \leq j \leq n-1}$ in $\mathscr{B}_n$, let

$$
U_r(b) = \sum_{t=1}^{n-r} (b_{t,t+r-1} - b_{t,t+r}) + \sum_{t=n-r+1}^{n-1} \chi\{b_{t,n-1} > n - t\}. \tag{1.1}
$$

Here $\chi\{\dots\}$ has value 1 when the statement "$\dots$" is true and 0 otherwise, and we use the convention that $b_{i,n} = n - i$ for all i and $b_{0,j} = n$ for all j.

Mills, Robbins and Rumsey conjectured that U_r has the same distribution as the position of the 1 in the top row of an alternating sign matrix, and presented several conjectures related to the distribution of the statistics U_r. The aim of this paper is to obtain the generating functions for the enumerations concerning these conjectures. In this introduction, we briefly review these conjectures by Mills, Robbins and Rumsey, and present a Pfaffian expression for each problem. In fact, we generalize the definition of $\mathscr{B}_n$ to $\mathscr{B}_{n,m}$ in Section 3, and consider the generating functions in wider classes of plane partitions, so that these results are special cases of the theorems obtained in the following sections. For the definition of the numbers A_n, A_n^k, $A_n^{k,l}$, $A_{n,r}^{\mathrm{VS}}$, $A_n^{\mathrm{VS},r}$ and the polynomials $A_n(t)$, $A_n(t,u)$, $A_{2n+1}^{\mathrm{VS}}(t)$, the reader should refer to the Section 2. It seems that these numbers have the standard notation which have appeared concerning the alternating sign matrices (see [23, 27, 29, 34]). Let $\bar{S}_n = (\bar{s}_{ij})_{1 \leq i,j \leq n}$ be the skew-symmetric matrix of size n whose (i,j)th entry $\bar{s}_{ij}$ is equal to $(-1)^{j-i-1}$ for $1 \leq i < j \leq n$, and let O_n denote the $n \times n$ zero matrix. Let $J_n = (\delta_{i,n+1-j})_{1 \leq i,j \leq n}$ denote the anti-diagonal matrix where $\delta_{i,j}$ stands for the Kronecker delta function. First of all, Mills, Robbins and Rumsey presented the following conjecture in the paper [26], which we call the refined enumeration of TSSCPPs:

Conjecture 1.1. ([26, pp.282, Conjecture 2]) Let n be a positive integer. Let $1 \leq k \leq n$ and $1 \leq r \leq n$. Then the number of elements b of $\mathscr{B}_n$ such that $U_r(b) = k - 1$ would be A_n^k. Namely, $\sum_{b \in \mathscr{B}_n} t^{U_r(b)} = A_n(t)$ would hold.

Let n and N be positive integers, and let $B_n^N(t) = (b_{ij}(t))_{0 \leq i \leq n-1,\, 0 \leq j \leq n+N-1}$ be the $n \times (n+N)$ matrix whose (i,j)th entry is

$$
b_{ij}(t) = \begin{cases} \delta_{0,j} & \text{if } i = 0, \\ \binom{i-1}{j-i} + \binom{i-1}{j-i-1}t & \text{otherwise.} \end{cases}
$$

Especially, when $t = 1$, we write B_n^N for $B_n^N(1)$ whose (i,j)th entry is $\binom{i}{j-i}$. One of the results we obtain for Conjecture 1.1 is following:

Theorem 1.2. Let n be a positive integer and let N be an even integer such that $N \geq n - 1$. Then

$$
\sum_{b \in \mathscr{B}_n} t^{U_r(b)} = \operatorname{Pf} \begin{pmatrix} O_n & J_n B_n^N(t) \\ -{}^t B_n^N(t) J_n & \bar{S}_{n+N} \end{pmatrix}. \tag{1.2}
$$

(cf. Corollary 3.6, Theorem 4.1, Corollary 7.5 and Corollary 8.3).

For example, if $n = 3$ and $N = 2$ then the above Pfaffian looks like as follows.

$$
\operatorname{Pf} \left(\begin{array}{ccc|ccccc} 0 & 0 & 0 & 0 & 0 & 1 & 1+t & t \\ 0 & 0 & 0 & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 1 & -1 & 1 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 \\ -1 & -t & 0 & 1 & -1 & 0 & 1 & -1 \\ -1-t & 0 & 0 & -1 & 1 & -1 & 0 & 1 \\ -t & 0 & 0 & 1 & -1 & 1 & -1 & 0 \end{array} \right).
$$

In the same paper, they also presented the following conjecture which we call the doubly refined enumeration of TSSCPPs:

Conjecture 1.3. ([26, pp.284, Conjecture 3], [34]) Let $n \geq 2$ and $1 \leq k, l \leq n$ be integers. Then the number of elements b of $\mathscr{B}_n$ such that $U_1(b) = k - 1$ and $U_2(b) = n - l$ would be $A_n^{k,l}$.

Let n and N be positive integers. Let $B_n^N(t, u) = (b_{ij}(t, u))_{0 \leq i \leq n-1,\, 0 \leq j \leq n+N-1}$ be the $n \times (n+N)$ matrix whose (i,j)th entry is

$$
b_{ij}(t, u) = \begin{cases} \delta_{0,j} & \text{if } i = 0, \\ \delta_{0,j-i} + \delta_{0,j-i-1}tu & \text{if } i = 1, \\ \binom{i-2}{j-i} + \binom{i-2}{j-i-1}(t+u) + \binom{i-2}{j-i-2}tu & \text{otherwise.} \end{cases}
$$

Note that, when $u = 1$, $B_n^N(t, 1)$ is equal to $B_n^N(t)$. Then one form of the Pfaffian expressions for Conjecture 1.3 which we obtain in this paper is following:

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Theorem 1.4. Let n be a positive integer and let N be an even integer such that $N \geq n - 1$. If r is an integer such that $2 \leq r \leq n$, then we have

$$\sum_{b \in \mathscr{B}_n} t^{U_1(b)} u^{U_r(b)} = \mathrm{Pf} \begin{pmatrix} O_n & J_n B_n^N(t, u) \\ -{}^t B_n^N(t, u) J_n & \bar{S}_{n+N} \end{pmatrix}. \tag{1.3}$$

(cf. Corollary 3.6, Theorem 4.1, Corollary 7.3 and Corollary 8.2).

The monotone triangles are known to be in one-to-one correspondence with the alternating sign matrices ([4, 25]). Here we arrange our definition following the notation in [26]. A *monotone triangle* of size n is, by definition, a triangular array of positive integers

$$\begin{array}{ccccc}
& & & & m_{n,n} \\
& & & m_{n-1,n-1} & m_{n-1,n} \\
& \cdot^{\cdot^{\cdot}} & & \vdots & \vdots \\
m_{1,1} & \cdots & & m_{1,n-1} & m_{1,n}
\end{array}$$

subject to the constraints that

(M1) $m_{ij} < m_{i,j+1}$ whenever both sides are defined,

(M2) $m_{ij} \geq m_{i+1,j}$ whenever both sides are defined,

(M3) $m_{ij} \leq m_{i+1,j+1}$ whenever both sides are defined,

(M4) the bottom row $(m_{1,1}, m_{1,2}, \ldots, m_{1,n})$ is $(1, 2, \ldots, n)$.

Let $\mathscr{M}_n$ denote the set of monotone triangles of size n. For example, $\mathscr{M}_3$ consists of the following seven elements.

$$\begin{array}{ccccccc}
1 & 2 & 1 & 2 & 3 & 2 & 3 \\
1 \ \ 2 & 1 \ \ 2 & 1 \ \ 3 & 1 \ \ 3 & 1 \ \ 3 & 2 \ \ 3 & 2 \ \ 3 \\
1 \ \ 2 \ \ 3 & 1 \ \ 2 \ \ 3 & 1 \ \ 2 \ \ 3 & 1 \ \ 2 \ \ 3 & 1 \ \ 2 \ \ 3 & 1 \ \ 2 \ \ 3 & 1 \ \ 2 \ \ 3
\end{array}$$

Note that, if one removes the bottom row of $m \in \mathscr{M}_n$ and turn it upside-down, then he get an array defined in [26].

For $k = 0, 1, \ldots, n - 1$, let $\mathscr{M}_n^k$ denote the set of monotone triangles with all entries m_{ij} in the first $n - k$ columns equal to their minimum values $j - i + 1$. For $k = 0, 1, \ldots, n - 1$, let $\mathscr{B}_n^k$ be the subset of those b in $\mathscr{B}_n$ such that all b_{ij} in the first $n - 1 - k$ columns are equal to their maximal values n. Then they also presented the following conjecture:

Conjecture 1.5. ([26, pp.287, Conjecture 7]) For $n \geq 2$ and $k = 0, 1, \ldots, n - 1$, the cardinality of $\mathscr{B}_n^k$ is equal to the cardinality of $\mathscr{M}_n^k$.

Let m, n and k be integers such that $1 \leq m \leq n$ and $0 \leq k \leq n - m$. We define the $n \times n$ skew-symmetric matrix $\bar{L}_n^{(m,k)}(\varepsilon) = (\bar{l}_{ij}^{(m,k)}(\varepsilon))_{1 \leq i,j \leq n}$ as follows: if k is even, then

$$\bar{l}_{ij}^{(m,k)}(\varepsilon) = \begin{cases} (-1)^{j-i-1}\varepsilon & \text{if } 1 \leq i < j \leq n \text{ and } i \leq m + k, \\ (-1)^{j-i-1} & \text{if } m + k < i < j \leq n, \end{cases}$$

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else

$$\bar{l}_{ij}^{(m,k)}(\varepsilon) = \begin{cases} (-1)^{j-i-1}\varepsilon & \text{if } 1 \le i < j \le m+k, \\ (-1)^{j-i-1} & \text{if } 1 \le i < j \le n \text{ and } m+k < j. \end{cases}$$

For example,

$$\bar{L}_6^{(2,1)}(\varepsilon) = \begin{pmatrix} 0 & \varepsilon & -\varepsilon & 1 & -1 & 1 \\ -\varepsilon & 0 & \varepsilon & -1 & 1 & -1 \\ \varepsilon & -\varepsilon & 0 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 1 & -1 & 0 \end{pmatrix}, \quad \bar{L}_6^{(2,2)}(\varepsilon) = \begin{pmatrix} 0 & \varepsilon & -\varepsilon & \varepsilon & -\varepsilon & \varepsilon \\ -\varepsilon & 0 & \varepsilon & -\varepsilon & \varepsilon & -\varepsilon \\ \varepsilon & -\varepsilon & 0 & \varepsilon & -\varepsilon & \varepsilon \\ -\varepsilon & \varepsilon & -\varepsilon & 0 & \varepsilon & -\varepsilon \\ \varepsilon & -\varepsilon & \varepsilon & -\varepsilon & 0 & 1 \\ -\varepsilon & \varepsilon & -\varepsilon & \varepsilon & -1 & 0 \end{pmatrix}.$$

Then a Pfaffian expression for Conjecture 1.5 which we obtain in this paper is following:

Theorem 1.6. Let n be a positive integer and let $k = 0, 1, \ldots, n-1$. Let N be an even integer such that $N \ge k$. The cardinality of $\mathscr{B}_n^k$ is equal to

$$\lim_{\varepsilon \to 0} \varepsilon^{-\lfloor \frac{k}{2} \rfloor} \operatorname{Pf} \begin{pmatrix} O_n & B_n^N J_{n+N} \\ -J_{n+N}{}^t B_n^N & \bar{L}_{n+N}^{(n,k)}(\varepsilon) \end{pmatrix} \tag{1.4}$$

Here $\lfloor x \rfloor$ stands for the floor function, i.e. the greatest integer less than or equal to x. (cf. Theorem 5.1, Conjecture 5.2, Corollary 7.8 and Corollary 8.4).

This paper is composed as follows. All through the paper, we consider more general set $\mathscr{B}_{n,m}$ (see Definition 3.2) of shifted plane partitions, which first appeared in [18, Theorem 1]. Thus the above theorems, which give Pfaffian expressions for the Mills, Robbins and Rumsey conjectures, are the special cases of the generating functions we obtain in this paper. The key idea is to construct a bijection between $\mathscr{B}_{n,m}$ and a new set $\mathscr{P}_{n,m}$ which is more easy to understand. In Section 3, we define this set $\mathscr{P}_{n,m}$ (see Definition 3.1) of ordinary plane partitions which is the main object we study throughout the paper. We establish a bijection between $\mathscr{P}_{n,m}$ and a set $\mathscr{T}_{n,m}$ of totally symmetric self-complementary plane partitions (see Theorem 3.5), and also construct a bijection between $\mathscr{B}_{n,m}$ and $\mathscr{T}_{n,m}$ (see Theorem 3.4). As a corollary we obtain a bijection between $\mathscr{P}_{n,m}$ and $\mathscr{B}_{n,m}$ (see Corollary 3.6) which makes it possible to interpret all the properties of $\mathscr{B}_n$ studied in [25] in the words of $\mathscr{P}_n$ (here we write $\mathscr{B}_n$ for $\mathscr{B}_{n,0}$ and $\mathscr{P}_n$ for $\mathscr{P}_{n,0}$). Thus we reduce the enumeration problems of the totally symmetric self-complementary plane partitions to the study of our new object $\mathscr{P}_{n,m}$, which we call "the restricted column-strict plane partitions". This reveals several mysterious properties of this new object $\mathscr{P}_{n,m}$ which resembles the classical theory of the tableaux and Schur functions. Before we proceed to Section 3, we collect some basic definitions and several fundamental theorems in Section 2. Especially the minor summation formula and its applications (see Theorem 2.1 and Proposition 2.3) will be an important tool to obtain the generating functions in Section 7. But the reader can skip this section now, and use it as a reference when he need it. In Section 4, we generalize the statistics (1.1) to the general set $\mathscr{B}_{n,m}$ (see (4.1)) and restate it as the statistics of $\mathscr{P}_{n,m}$ (see Theorem 4.1). We also present a new conjecture (Conjecture 4.2) which is not in

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[25]. In Section 5, we restate Conjecture 1.5 in the words of $\mathscr{P}_n$ (Theorem 5.1) and also present a new conjecture (Conjecture 5.2) which is a refined version of Conjecture 1.5. We also give a restatement of [25, Conjecture 7'] in the words of $\mathscr{P}_n$ (Theorem 5.4). In Section 6, we translate the strange enumeration (especially (-1)-enumeration) of the totally symmetric self-complementary plane partitions in the words of $\mathscr{P}_{n,m}$ (see Theorem 6.1). Then Section 7 contains the main results of this paper, i.e. we obtain several generating functions concerning $\mathscr{P}_{n,m}$ using the lattice paths. We give Corollary 7.5 for the refined TSSCPP enumeration, Corollary 7.3 for the doubly refined TSSCPP enumeration, and Corollary 7.8 for Conjecture 1.5. In Section 8 we give a constant term identity for each Pfaffian obtained in Section 7 (see Corollary 8.2, Corollary 8.3 and Corollary 8.4). Each of the constant identities includes a different Littlewood type identity. This reveals a certain relation between the Littlewood type identity for the enumerations of the Schur functions and the enumerations of the TSSCPPs.

2 Preliminaries

First we recall the numbers and polynomials related to the alternating sign matrices (cf. [4, 23, 25, 27, 28, 29, 34, 36]). In the latter half of this section, we recall the notation of partitions and the results on Pfaffians which will be needed in the following sections, i.e. Theorem 2.1 and Proposition 2.3. For the details on partitions the reader is referred to [24, 31], and for the explanation on Pfaffians the reader can consult [15, 16, 17]. Let A_n denote the number defined by

$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}. \tag{2.1}$$

This number is famous for the alternating sign matrix conjecture (cf. [4]). The number of totally symmetric self-complementary plane partitions was conjectured to be A_n in [26, pp.282, Conjecture 1], and settled in [32, p.p.127, Theorem 8.3] and [2] (see also [1, 3]), Another proof was appeared in [18] and several determinant techniques have been developed in it. Let n be a positive number and let $1 \leq r \leq n$. Set A_n^r to be the number

$$A_n^r = \frac{\binom{n+r-2}{n-1}\binom{2n-r-1}{n-1}}{\binom{2n-2}{n-1}} A_{n-1} = \frac{\binom{n+r-2}{n-1}\binom{2n-1-r}{n-1}}{\binom{3n-2}{n-1}} A_n. \tag{2.2}$$

Then the number A_n^r satisfies the recurrence $A_n^1 = A_{n-1}$ and

$$\frac{A_n^{r+1}}{A_n^r} = \frac{(n-r)(n+r-1)}{r(2n-r-1)}.$$

The number has appeared to describe the distribution of the position of the 1 in the top row of an alternating sign matrix (see [21, 23, 27, 36]). We also define the polynomial $A_n(t) = \sum_{r=1}^n A_n^r t^{r-1}$. For instance, the first few terms are $A_1(t) = 1$, $A_2(t) = 1 + t$, $A_3(t) = 2 + 3t + 2t^2$, $A_4(t) = 7 + 14t + 14t^2 + 7t^3$. Let n be a

positive integer and let $A_n^{k,l}$, $1 \leq k, l \leq n$, denote the number which satisfies the initial condition

$$A_n^{k,1} = A_n^{1,k} = \begin{cases} 0 & \text{if } k = 1 \\ A_{n-1}^{k-1} & \text{if } 2 \leq k \leq n \end{cases} \tag{2.3}$$

and the recurrence equation

$$A_n^{k+1,l+1} - A_n^{k,l} = \frac{A_{n-1}^k(A_n^{l+1} - A_n^l) + A_{n-1}^l(A_n^{k+1} - A_n^k)}{A_n^1} \tag{2.4}$$

for $1 \leq k, l \leq n - 1$. This recurrence equation satisfied by $A_n^{k,l}$ has been introduced by Stroganov in [34, Section 5] to describe the double distribution of the positions of the 1's in the top row and the bottom row of an alternating sign matrix. For example, if $n = 3, 4$, then we have

$$\left(A_3^{k,l} \right)_{1 \leq k, l \leq 3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad \left(A_4^{k,l} \right)_{1 \leq k, l \leq 4} = \begin{pmatrix} 0 & 2 & 3 & 2 \\ 2 & 4 & 5 & 3 \\ 3 & 5 & 4 & 2 \\ 2 & 3 & 2 & 0 \end{pmatrix}.$$

Let $A_n(t, u)$ denote the polynomial defined by $A_n(t, u) = \sum_{k,l=1}^n A_n^{k,l} t^{k-1} u^{n-l}$. Thus we have $A_3(t, u) = 1 + t + u + tu + t^2u + tu^2 + t^2u^2$. Let $\omega = e^{2i\pi/3}$. Di Francesco and Zinn-Justin showed that $A_n(t, u)$ can be expressed by the Schur function as

$$A_n(t, u) = \frac{\{\omega^2(\omega + t)(\omega + u)\}^{n-1}}{3^{n(n-1)/2}} s_{\delta(n-1,n-1)}^{(2n)} \left(\frac{1 + \omega t}{\omega + t}, \frac{1 + \omega u}{\omega + u}, 1, \ldots, 1 \right) \tag{2.5}$$

where $s_\lambda^{(n)}(x_1, \ldots, x_n)$ stands for the Schur function in the n variables x_1, $\ldots$, x_n, corresponding to the partition λ, and $\delta(n-1, n-1) = (n-1, n-1, n-2, n-2, \ldots, 1, 1)$ (See [7, pp.4], [27]).

Let A_{2n+1}^{VS} be the number defined by

$$A_{2n+1}^{\mathrm{VS}} = (-3)^{n^2} \prod_{\substack{1 \leq i, j \leq 2n+1 \\ 2 | j}} \frac{3(j - i) + 1}{j - i + 2n + 1} = \frac{1}{2^n} \prod_{k=1}^n \frac{(6k - 2)!(2k - 1)!}{(4k - 1)!(4k - 2)!} \tag{2.6}$$

and let $A_{2n+1}^{\mathrm{VS},\, r}$ be the number given by

$$A_{2n+1}^{\mathrm{VS},\, r} = \frac{A_{2n-1}^{\mathrm{VS}}}{(4n - 2)!} \sum_{k=1}^r (-1)^{r+k} \frac{(2n + k - 2)!(4n - k - 1)!}{(k - 1)!(2n - k)!}. \tag{2.7}$$

This number A_{2n+1}^{VS} is equal to the number of vertically symmetric alternating sign matrices of size $2n + 1$ (see [23, 27, 29]). For example, the first few terms of A_{2n+1}^{VS} are 1, 3, 26, 646 and 45885. We also define the polynomial $A_{2n+1}^{\mathrm{VS}}(t)$ by

$$A_{2n+1}^{\mathrm{VS}}(t) = \sum_{r=1}^{2n} A_{2n+1}^{\mathrm{VS},\, r} t^{r-1}. \tag{2.8}$$

For instance, the first few terms of (2.8) are $A_3^{\mathrm{VS}}(t) = 1$, $A_5^{\mathrm{VS}}(t) = 1 + t + t^2$, $A_7^{\mathrm{VS}}(t) = 3 + 6t + 8t^2 + 6t^3 + 3t^4$ and $A_9^{\mathrm{VS}}(t) = 26 + 78t + 138t^2 + 162t^3 + 138t^4 + 78t^5 + 26t^6$.

Next we recall and fix the notation of partitions and shifted partitions. We follow the notation and terminology of Macdonald [24]. Let $\mathbb{P}$ denote the set of positive integers. A partition is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers in non-increasing order: $\lambda_1 \geq \lambda_2 \geq \dots$ and containing only finitely many non-zero terms. The non-zero λ_i are call the *parts* of λ. The number of parts is the *length* of λ, denoted by $\ell(\lambda)$; and the sum of parts is the *weight* of λ, denoted by $|\lambda|$. The *diagram* of a partition λ may be formally defined as the set of lattice points $(i, j) \in \mathbb{P}^2$ such that $1 \leq j \leq \lambda_i$. We identify λ with its diagram. The number of nodes in the main diagonal of the diagram is called the *diagonal length* $d(\lambda) = \sharp\{i : \lambda_i \geq i\}$. The *conjugate* of a partition λ is the partition λ' whose diagram is the transpose of the diagram of λ. A *self-conjugate partition* is a partition whose conjugate partition is equal to itself. For a partition λ, let $r(\lambda)$ denote the number of rows of odd length, and let $c(\lambda)$ denote the number of columns of odd length.

A partition with distinct parts is called a *strict partition*. The *shifted diagram* of a strict partition μ is the set of lattice points $(i, j) \in \mathbb{P}^2$ such that $i \leq j \leq \mu_i + i$. We identify a strict partition with its shifted diagram. If λ is a self-conjugate partition, we can associate a strict partition $(\mu_i)_{1 \leq i \leq d(\lambda)}$ to λ where $\mu_i = \lambda_i - i + 1$, and this defines a bijection of the set of self-conjugate partitions onto the set of strict partitions. We say that a partition is *even* if all of its parts λ_i are even.

A *q-binomial coefficient* is, by definition, $\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{(q)_n}{(q)_r (q)_{n-r}}$, where $(q)_k = \prod_{i=1}^{k}(1 - q^i)$. A binomial coefficient is written as $\binom{n}{r} = \begin{bmatrix} n \\ r \end{bmatrix}_1$. For any finite set S and a non-negative integer r, let $\binom{S}{r}$ denote the set of all r-element subsets of S. For a subset $I = \{i_1, \dots, i_r\} \in \binom{[n]}{r}$, let $\overline{I}$ denote the set-theoretic complement of I in $[n]$. Let m, n and r be integers such that $r \leq m, n$ and let T be an m by n matrix. For any index sets $I = \{i_1, \dots, i_r\} \in \binom{[m]}{r}$ and $J = \{j_1, \dots, j_r\} \in \binom{[n]}{r}$, let A_J^I denote the submatrix obtained by selecting the rows indexed by I and the columns indexed by J. If $r = m$ and $I = [m]$, we simply write A_J for $A_J^{[m]}$. Similarly, if $r = n$ and $J = [n]$, we write A^I for $A_{[n]}^I$. The following identity (2.9) follows from the proof of [16, Theorem 4.2]) which we call the minor summation formula here (see also [10]). The formula is equivalent to the even case (2.12) and the odd case (2.14) which appeared in [15], but the merit of using (2.9) is that it does not depend on whether n is even or odd.

Theorem 2.1. Let m and n be positive integers such that $m \leq n$ and $n - m$ is even. Let $T = (t_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be an m by n rectangular matrix. Let B be a skew-symmetric matrix of size n. Then

$$\sum_{I \in \binom{[n]}{m}} (-1)^{s(\overline{I}, I)} \mathrm{Pf}(B_{\overline{I}}^{\overline{I}}) \det(T_I) = \mathrm{Pf} \begin{pmatrix} O_m & T J_n \\ -J_n {}^t T & J_n {}^t B J_n \end{pmatrix} = \mathrm{Pf} \begin{pmatrix} O_m & J_m T \\ -{}^t T J_m & B \end{pmatrix}.$$

(2.9)

Here $\overline{I} = [n] \setminus I$, and $s(\overline{I}, I)$ denote the shuffle number to merge $\overline{I}$ with I into $[n]$.

From here we define several skew-symmetric matrices which play an important role in the applications. Let n be a positive integer. Let $S_n = (s_{ij})_{1 \leq i, j \leq n}$ be the skew-symmetric matrix of size n whose (i, j)th entry s_{ij} is 1 for $1 \leq i < j \leq n$, and let $\bar{S}_n$ be as defined in Section 1. Let $\mathrm{rem}(a, b)$ denote the remainder of

a divided by b, and let t be an indeterminate. Let the $n \times n$ skew-symmetric matrices $R_n(t) = (r_{ij}(t))_{1 \leq i,j \leq n}$, $C_n(t) = (c_{ij}(t))_{1 \leq i,j \leq n}$, $\bar{R}_n(t) = (\bar{r}_{ij}(t))_{1 \leq i,j \leq n}$ and $\bar{C}_n(t) = (\bar{c}_{ij}(t))_{1 \leq i,j \leq n}$ be defined as $r_{ij}(t) = t^{\mathrm{rem}(i-1,2)+\mathrm{rem}(j,2)}$, $c_{ij}(t) = t^{j-i-1}$, $\bar{r}_{ij}(t) = (-1)^{j-i-1} t^{j-i-1}$ and $\bar{c}_{ij}(t) = (-1)^{j-i-1} t^{\mathrm{rem}(n+1-i,2)+\mathrm{rem}(n-j,2)}$ for $1 \leq i < j \leq n$, respectively. Further we write R_n, C_n, $\bar{R}_n$ and $\bar{C}_n$ for $R_n(0)$, $C_n(0)$, $\bar{R}_n(0)$ and $\bar{C}_n(0)$, respectively. For example,

$$R_4(t) = \begin{pmatrix} 0 & 1 & t & 1 \\ -1 & 0 & t^2 & t \\ -t & -t^2 & 0 & 1 \\ -1 & -t & -1 & 0 \end{pmatrix} \quad \text{and} \quad C_4(t) = \begin{pmatrix} 0 & 1 & t & t^2 \\ -1 & 0 & 1 & t \\ -t & -1 & 0 & 1 \\ -t^2 & -t & -1 & 0 \end{pmatrix}.$$

Let m, n and k be integers such that $1 \leq m \leq n$ and $0 \leq k \leq n-m$, and let ε be an indeterminate. Let $\bar{L}_n^{(m,k)}(\varepsilon)$ be as in Section 1, and let $L_n^{(m,k)}(\varepsilon) = (l_{ij}^{(m,k)}(\varepsilon))_{1 \leq i,j \leq n}$ denote the $n \times n$ skew-symmetric matrix whose (i,j)th entry is

$$l_{ij}^{(m,k)}(\varepsilon) = \begin{cases} 1 & \text{if } 1 \leq i < j \leq m+k, \\ \varepsilon & \text{if } 1 \leq i < j \leq n \text{ and } m+k < j. \end{cases}$$

The following lemma (cf. [15, Section 4, Lemma 7]) is very useful to compute the subpfaffians of a given skew-matrix, and the basic idea to prove Proposition 2.3.

Lemma 2.2. Let x_i and y_j be indeterminates, and let n is a non-negative integer. Then

$$\mathrm{Pf}\,[x_i y_j]_{1 \leq i < j \leq 2n} = \prod_{i=1}^{n} x_{2i-1} \prod_{i=1}^{n} y_{2i}. \quad \square$$

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition such that $\ell(\lambda) \leq m$. Let $I_m(\lambda)$ denote the m-element set $\{\lambda_m + 1, \lambda_{m-1} + 2, \ldots, \lambda_1 + m\}$. For example, if $m = 4$ and $\lambda = (4,3,1)$, then $I_4(\lambda) = \{1,3,6,8\}$. We use this notation to fix a row/column index set of a given matrix. The following proposition is useful in combination with Theorem 2.1:

Proposition 2.3. Let m and n be positive integers such that $m \leq n$. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition such that $\ell(\lambda) \leq m$ and $\lambda_1 = \ell(\lambda') \leq n-m$. Here $\bar{I} = [n] \setminus I$ denote the complement in the set $[n]$.

(i) If m or $n-m$ is even then we have $(-1)^{s(\overline{I_m(\lambda)}, I_m(\lambda))} = (-1)^{s(I_m(\lambda), \overline{I_m(\lambda)})} = (-1)^{|\lambda|}$.

(ii) Assume m is even and $0 \leq k \leq n-m$. Then we have $\mathrm{Pf}\left(C_n(t)_{I_m(\lambda)}^{I_m(\lambda)}\right) = t^{c(\lambda)}$ and $\mathrm{Pf}\left(R_n(t)_{I_m(\lambda)}^{I_m(\lambda)}\right) = t^{r(\lambda)}$. In particular, we have $\mathrm{Pf}\left(S_{n \, I_m(\lambda)}^{I_m(\lambda)}\right) = 1$ for any λ, and $\mathrm{Pf}\left(R_{n \, I_m(\lambda)}^{I_m(\lambda)}\right)$ (resp. $\mathrm{Pf}\left(C_{n \, I_m(\lambda)}^{I_m(\lambda)}\right)$) equals 1 if all rows (resp. columns) of λ have even length, and 0 otherwise. Further, $\lim_{\varepsilon \to 0} \mathrm{Pf}\left(L_n^{(m,k)}(\varepsilon)_{I_m(\lambda)}^{I_m(\lambda)}\right)$ equals 1 if $\lambda_1 \leq k$, and 0 otherwise.

(iii) Assume $n-m$ is even and $0 \leq k \leq n-m$. Then, we have $\mathrm{Pf}\left(\bar{C}_n(t)_{\overline{I_m(\lambda)}}^{\overline{I_m(\lambda)}}\right) = (-1)^{|\lambda|} t^{c(\lambda)}$ and $\mathrm{Pf}\left(\bar{R}_n(t)_{\overline{I_m(\lambda)}}^{\overline{I_m(\lambda)}}\right) = (-1)^{|\lambda|} t^{r(\lambda)}$. In particular, we have $\mathrm{Pf}\left(\bar{S}_{n}^{\overline{I_m(\lambda)}}_{\overline{I_m(\lambda)}}\right) =$

$(-1)^{|\lambda|}$ for any λ, and $\mathrm{Pf}\left(R_n \overline{\frac{I_m(\lambda)}{I_m(\lambda)}}\right)$ (resp. $\mathrm{Pf}\left(C_n \overline{\frac{I_m(\lambda)}{I_m(\lambda)}}\right)$) equals $(-1)^{|\lambda|}$ if all rows (resp. columns) of λ have even length, and 0 otherwise. Further, $\lim_{\varepsilon \to 0} \varepsilon^{-\lfloor \frac{k}{2} \rfloor} \mathrm{Pf}\left(\bar{L}_n^{(m,k)}(\varepsilon) \overline{\frac{I_m(\lambda)}{I_m(\lambda)}}\right)$ equals $(-1)^{|\lambda|}$ if $\lambda_1 \le k$, and 0 otherwise.

We can also use Lemma 2.2 to prove this proposition. But the important part is that we can combine Proposition 2.3 with Theorem 2.1 to compute several sums of determinants. For example, if we take $B = \bar{R}_n(t)$ in (2.9), then we obtain

$$\sum_{\substack{\lambda \\ \ell(\lambda) \le m, \ \ell(\lambda') \le n-m}} t^{r(\lambda)} \det(T_{I_m(\lambda)}) = \mathrm{Pf}\begin{pmatrix} O_m & J_m T \\ -{}^t T J_m & \bar{R}_n(t) \end{pmatrix}$$

for any positive integers $n \ge m$. If we put $t = 1$ in the left-hand side, then we obtain the sum of $\det(T_{I_m(\lambda)})$ where λ runs over all partitions, meanwhile, if we put $t = 0$, then we obtain the sum of $\det(T_{I_m(\lambda)})$ where λ runs over all even partitions.

In [16] we defined the notion of copfaffian matrices. Let n be an even integer, and let A be a skew symmetric matrix of size n. Thus $A\overline{\frac{\{i,j\}}{\{i,j\}}}$ denote the $(n-2)$ by $(n-2)$ skew symmetric sub-matrix obtained by removing both the ith and jth rows and both the ith and jth columns of A for $1 \le i < j \le n$. Let us define $\gamma(i,j)$ by

$$\gamma(i,j) = (-1)^{j-i-1} \mathrm{Pf}\left(A\overline{\frac{\{i,j\}}{\{i,j\}}}\right) \tag{2.10}$$

for $1 \le i < j \le n$. We define the values of $\gamma(i,j)$ for $1 \le j \le i \le n$ so that $\gamma(j,i) = -\gamma(i,j)$ always holds. Let n be an even integer. Given a skew symmetric matrix A of size n, let us call $\gamma(i,j)$ a *copfaffian* corresponding to a_{ij} (or (i,j)-*copfaffian*), and let $\hat{A}$ denote the skew symmetric matrix whose (i,j)th entry is $\gamma(i,j)$, which we call the *copfaffian matrix* of A. From Lemma 2.2, we can see that, when n is even, $\bar{S}_n$, $\bar{R}_n(t)$ and $\bar{C}_n(t)$ are the copfaffian matrices of S_n, $R_n(t)$ and $C_n(t)$ respectively, and vise versa.

In the following, we deduce another form (2.12), (2.14) of the minor summation formula (2.9) which appeared in [15]. Let $n = 2n'$ be an even integer and let A be a non-singular $n \times n$ skew symmetric matrix. In [16, Theorem 2.6], we have obtained the fact that, for any $I \in \binom{[n]}{2r}$,

$$\mathrm{Pf}\left(\hat{A}_I^I\right) = (-1)^{s(\bar{I},I)} \mathrm{Pf}(A)^{r-1} \mathrm{Pf}\left(A_{\bar{I}}^{\bar{I}}\right). \tag{2.11}$$

In particular, (2.11) implies $\mathrm{Pf}(\hat{A}) = \mathrm{Pf}(A)^{n'-1}$ and $\hat{\hat{A}} = \mathrm{Pf}(A)^{n'-2} A$. If we take $B = \frac{1}{\mathrm{Pf}(A)} \hat{A}$ in (2.9), (2.11) also shows that $(-1)^{s(I,\bar{I})} \mathrm{Pf}(B_{\bar{I}}^{\bar{I}}) = \frac{1}{\mathrm{Pf}(A)} \mathrm{Pf}(A_I^I)$. From this, one can deduce that

$$\sum_{I \in \binom{[n]}{m}} \mathrm{Pf}(A_I^I) \det(T_I) = \mathrm{Pf}(A) \mathrm{Pf}\begin{pmatrix} O_m & J_m T \\ -{}^t T J_m & \frac{1}{\mathrm{Pf}(A)} \hat{A} \end{pmatrix} = \mathrm{Pf}(Q_{ij})_{1 \le i,j \le n}, \tag{2.12}$$

where $Q = (Q_{ij}) = T A \, {}^t T$, and its entries are given by

$$Q_{ij} = \sum_{1 \le k < l \le N} a_{kl} \det(T_{kl}^{ij}), \qquad (1 \le i, j \le m). \tag{2.13}$$

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Here we write T_{kl}^{ij} for $T_{\{kl\}}^{\{ij\}}$.

Assume m and $n = 2n' - 1$ is an odd integer and let $A_0 = (\alpha_{ij})_{0 \leq i,j \leq n}$ be a non-singular $(n+1) \times (n+1)$ skew symmetric matrix, and let $\hat{A}_0$ denote its copfaffian matrix. Let T be an $m \times n$ matrix. Write the $(n+1) \times (n+1)$ matrix $B_0 = \frac{1}{\text{Pf}(A_0)} \hat{A}_0 = \left(\begin{array}{c|c} 1 & b \\ \hline -{}^t b & B \end{array} \right)$ with $n \times n$ matrix B, and the $(m+1) \times (n+1)$ matrix $T^0 = \left(\begin{array}{c|c} 1 & O_{1,n} \\ \hline O_{m,1} & T \end{array} \right)$. Then apply (2.12) and we obtain

$$\sum_{I \in \binom{[n]}{m}} \text{Pf}(A^0{}_{\{0\} \uplus I}^{\{0\} \uplus I}) \det(T_I) = \text{Pf}(A^0) \text{Pf} \begin{pmatrix} O_m & J_m T \\ -{}^t T J_m & B \end{pmatrix} = \text{Pf}(Q_{ij})_{0 \leq i,j \leq n}, \quad (2.14)$$

where $Q_{0j} = \sum_{1 \leq k \leq n} a_{0k} t_{jk}$ (see also [15, Theorem 1]).

3 Definitions and bijections

In this section we study three classes of (shifted) plane partitions which are denoted by $\mathscr{T}_{n,m}$, $\mathscr{B}_{n,m}$ and $\mathscr{P}_{n,m}$, and we establish bijections between them. The set $\mathscr{B}_{n,m}$ is a generalization of the set $\mathscr{B}_n$ defined in [26], and the set $\mathscr{P}_{n,m}$ is newly defined in this paper. Thus the study of the totally symmetric plane partitions reduce to the study of $\mathscr{P}_{n,m}$, which we call the set of "restricted column-strict plane partitions" (see Definition 3.1). In the later sections, we intensively study the set $\mathscr{P}_{n,m}$ which enable us to reveal the several interesting properties of this set of plane partitions. First of all we have to recall the basic definitions and notation concerning plane partitions. For the general theory of plane partitions the reader may consult [4, 24, 30, 31, 32].

A *plane partition* is an array $\pi = (\pi_{ij})_{i,j \geq 1}$ of nonnegative integers such that π has finite support (i.e. finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum_{i,j \geq 1} \pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n, or π has *weight* n. A *part* of a plane partition $\pi = (\pi_{ij})_{i,j \geq 1}$ is a positive entry $\pi_{ij} > 0$. The *shape* of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row. We denote the shape of π by $\text{sh}(\pi)$. We also say that π has r *rows* if $r = \ell(\lambda)$. Similarly, π has s *columns* if $s = \ell(\lambda')$. A plane partition is said to be *column-strict* if it is strictly decreasing in columns. For example,

$$
\begin{array}{cccccc}
5 & 5 & 4 & 3 & 2 & 1 \\
4 & 4 & 2 & 2 & 1 \\
2 & 2 & 1 & 1 \\
1
\end{array}
$$

is a column-strict plane partition and has shape $(6, 5, 4, 1)$, 4 rows, 6 columns and weight 40. Consider the elements of $\mathbb{P}^3$, regarded as the lattice points of $\mathbb{R}^3$ in the positive orthant. The *Ferrers graph* $F(\pi)$ of π is the set of all lattice points $(i, j, k) \in \mathbb{P}^3$ such that $k \leq \pi_{ij}$. A subset F of $\mathbb{P}^3$ is a Ferrers graph if and only if it satisfies

$$x_1 \leq x_2, \ y_1 \leq y_2, \ z_1 \leq z_2 \text{ and } (x_2, y_2, z_2) \in F \Rightarrow (x_1, y_1, z_1) \in F.$$

Hereafter we identify a plane partition and its Ferrers graph, and write π for $F(\pi)$. The symmetric group S_3 is acting on $\mathbb{P}^3$ as permutations of the coordinate axes. A plane partition is said to be *totally symmetric* if its Ferrers graph is mapped to itself under all 6 permutations in S_3.

In this section we mainly consider three classes, i.e. $\mathscr{T}_{n,m}$, $\mathscr{B}_{n,m}$ and $\mathscr{P}_{n,m}$, of (shifted) plane partitions and construct bijections between them. First of all, the following set $\mathscr{P}_{n,m}$ of plane partitions plays a crucial role throughout this paper in the study of the totally symmetric plane partitions.

Definition 3.1. Let m and $n \geq 1$ be nonnegative integers. Let $\mathscr{P}_{n,m}$ denote the set of plane partitions $c = (c_{ij})_{1 \leq i,j}$ subject to the constraints that

(C1) c has at most n columns;

(C2) c is column-strict and each part in the jth column does not exceed $n + m - j$.

We call an element of $\mathscr{P}_{n,m}$ a *restricted column-strict plane partition* (abbreviated to RCSPP). When $m = 0$, we write $\mathscr{P}_n$ for $\mathscr{P}_{n,0}$. If a part in the jth column of c is equal to $n + m - j$ (that can happen only in the first row, i.e. $c_{1j} = n + m - j$), we call the part a *saturated part*. Further we define two subclasses of $\mathscr{P}_{n,m}$. Let $\mathscr{P}_{n,m}^{\mathrm{R}}$ denote the set of plane partitions c in $\mathscr{P}_{n,m}$ where each row has even length. and let $\mathscr{P}_{n,m}^{\mathrm{C}}$ denote the set of plane partitions c in $\mathscr{P}_{n,m}$ with each column of even length. We also write $\mathscr{P}_n^{\mathrm{R}}$ (resp. $\mathscr{P}_n^{\mathrm{C}}$) for $\mathscr{P}_{n,0}^{\mathrm{R}}$ (resp. $\mathscr{P}_{n,0}^{\mathrm{C}}$).

For instance, $\mathscr{P}_{1,2}$ consists of the following 4 plane partitions:

$$\emptyset \qquad 1 \qquad \mathbf{2} \qquad \begin{matrix} \mathbf{2} \\ 1 \end{matrix}$$

In the above four RCSPPs the boldfaced letters $\mathbf{2}$ stand for saturated parts since they are in the first column and equals $n + m - 1 = 2$.

Next we define the following set $\mathscr{B}_{n,m}$ of shifted plane partitions which is a generalization of $\mathscr{B}_n$ defined in [26, pp.281]. Let μ be a strict partition. A *shifted plane partition* τ of *shifted shape* μ is an arbitrary filling of the cells of μ with nonnegative integers such that each entry is weakly decreasing in rows and columns. In this paper we allow parts to be zero for shifted plane partitions of a fixed shifted shape μ.

Definition 3.2. (See [18, Theorem 1]). Let m and $n \geq 1$ be nonnegative integers. Let $\mathscr{B}_{n,m}$ denote the set of shifted plane partitions $b = (b_{ij})_{1 \leq i \leq j}$ subject to the constraints that

(B1) the shifted shape of b is $(n + m - 1, n + m - 2, \ldots, 2, 1)$;

(B2) $\max\{n - i, 0\} \leq b_{ij} \leq n$ for $1 \leq i \leq j \leq n + m - 1$.

When $m = 0$, we write $\mathscr{B}_n$ for $\mathscr{B}_{n,0}$. In this paper we call an element of $\mathscr{B}_{n,m}$ a *triangular shifted plane partition* (abbreviated to TSPP).

When $n = 1$ and $m = 2$, $\mathscr{B}_{1,2}$ consists of the following 4 elements:

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline \multicolumn{1}{|c|}{1} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \multicolumn{1}{|c|}{0} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|} \hline 1 & 0 \\ \hline \multicolumn{1}{|c|}{0} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|} \hline 0 & 0 \\ \hline \multicolumn{1}{|c|}{0} \\ \cline{1-1} \end{array}$$

The final object $\mathscr{T}_{n,m}$ we need to define in this section is a subclass of totally symmetric self-complementary plane partitions. Before we proceed to the definition of $\mathscr{T}_{n,m}$, we need to define additional terminology and symbols concerning plane partitions. Let $X_{r,s,t} = [r] \times [s] \times [t]$ denote the $r \times s \times t$ box. Assume r, s and t are all even. We divide this box into the eight regions $X_{r,s,t}^{+++}$, $X_{r,s,t}^{++-}$, $X_{r,s,t}^{+-+}$, $X_{r,s,t}^{+--}$, $X_{r,s,t}^{-++}$, $X_{r,s,t}^{-+-}$, $X_{r,s,t}^{--+}$ and $X_{r,s,t}^{---}$ depending on each of $x - r/2$, $y - s/2$ and $z - t/2$ is plus (> 0) or minus (≤ 0). For example $X_{r,s,t}^{-+-} = [1, r/2] \times [s/2 + 1, s] \times [1, t/2]$. Further we use the notation $X_{r,s,t}^{+} = X_{r,s,t}^{+++} \uplus X_{r,s,t}^{++-} \uplus X_{r,s,t}^{+-+} \uplus X_{r,s,t}^{-++}$ and $X_{r,s,t}^{-} = X_{r,s,t}^{+--} \uplus X_{r,s,t}^{-+-} \uplus X_{r,s,t}^{--+} \uplus X_{r,s,t}^{---}$. More generally we write $X_{r,s,t}(a, b, c) = [a - r/2 + 1, a + r/2] \times [b - s/2 + 1, b + s/2] \times [c - t/2 + 1, c + t/2]$ for the $r \times s \times t$ box centered at (a, b, c). We also use the notation $X_{r,s,t}^{\pm\pm\pm}(a, b, c)$ as the same meaning as above where each stands for one of the eight regions of $X_{r,s,t}(a, b, c)$. For example $X_{r,s,t}^{+-+}(a, b, c) = [a + 1, a + r/2] \times [b - s/2 + 1, b] \times [c + 1, c + t/2]$. The symbols $X_{r,s,t}^{\pm}(a, b, c)$ should be defined similarly. The involution $\sigma_{r,s,t} : (x, y, z) \mapsto (r + 1 - x, s + 1 - y, t + 1 - z)$ is called the *complementation*. When $r = s = t$ and $a = b = c$, we use the abbreviation X_r for $X_{r,r,r}$, $X_r^{\pm\pm\pm}$ for $X_{r,r,r}^{\pm\pm\pm}$, $X_r^{\pm}$ for $X_{r,r,r}^{\pm}$ and σ_r for $\sigma_{r,r,r}$. The symbols $X_r(a)$, $X_r^{\pm\pm\pm}(a)$ and $X_r^{\pm}(a)$ should be interpreted similarly.

A plane partition $\pi \subseteq X_{r,s,t}$ is (r, s, t)-self-complementary if we have, for all $p \in X_{r,s,t}$, $p \in \pi$ if and only if $\sigma_{r,s,t}(p) \notin \pi$. Let $\mathscr{T}_n$ denote the set of all plane partitions which is contained in the cube X_{2n}, $(2n, 2n, 2n)$-self-complementary and totally symmetric.

Definition 3.3. An element of $\mathscr{T}_n$ is called a totally symmetric self-complementary plane partition (abbreviated to TSSCPP) of size n. For nonnegative integers m and $n \geq 1$, let $\mathscr{T}_{n,m}$ denote the set of TSSCPPs $\pi \in \mathscr{T}_{n+m}$ of size $(n + m)$ which satisfy

(T) each $p \in \pi \cap X_{2m}(n)$ must be contained in $X_{2(n+m)}^{-}$.

Note that, when $\pi \in \mathscr{T}_{n,m}$, $\pi \cap X_{2m}(n)$ is uniquely determined by the condition (T), i.e. $\pi \cap X_{2m}(n) = X_{2m}^{-}(n)$.

For instance $\mathscr{T}_{1,2}$ is composed of four elements which are designated in Figure 1.

Mills, Robbins and Rumsey have constructed a bijection between $\mathscr{T}_n$ and $\mathscr{B}_n$ (see [26, Theorem 1]). The set $\mathscr{B}_{n,m}$ is first considered in [18, Theorem 1 (2)]. The main results of this section is the bijections between the three classes $\mathscr{T}_{n,m}$, $\mathscr{B}_{n,m}$ and $\mathscr{P}_{n,m}$. Thus, in the later sections, we intensively study the properties of $\mathscr{P}_{n,m}$. The first theorem establish a bijection between $\mathscr{T}_{n,m}$ and $\mathscr{B}_{n,m}$, and this bijection is a generalization of the bijection between $\mathscr{T}_n$ and $\mathscr{B}_n$ constructed in [26, Theorem 1].

Theorem 3.4. Let m and $n \geq 1$ be nonnegative integers and (a_{ij}) be a TSSCPP in $\mathscr{T}_{n,m}$. Associate to the array (a_{ij}) the triangular array $a_{i+1,j+1} - (n + 2m)$ with $1 \leq i \leq j \leq n + m - 1$. Then the triangular array is in $\mathscr{B}_{n,m}$ and this mapping from $\mathscr{T}_{n,m}$ to $\mathscr{B}_{n,m}$ is a bijection.

In the set-theoretic approach this bijection $\mathscr{T}_{n,m} \to \mathscr{B}_{n,m}$, $a \mapsto b$ can be restated as follows:

$$\forall (x, y, z) \in \mathbb{P}^3 \text{ such that } 1 \leq x \leq n + m - 1:$$
$$(x, y, z) \in b \Leftrightarrow (x + 1, y + 1, z + n + 2m) \in a. \tag{3.1}$$

The second theorem establish a bijection between $\mathscr{T}_{n,m}$ and $\mathscr{P}_{n,m}$,

Theorem 3.5. Let m and $n \geq 1$ be nonnegative integers and $a = (a_{ij})_{1 \leq i,j \leq 2(n+m)}$ be a TSSCPP in $\mathscr{T}_{n,m}$. Associate to the array a the array $a_{i+n+m,j} - (n + m) - i + 1$ with $1 \leq i, j \leq n + m$. Then, by ignoring the zeros and negative entries, this array is a plane partition in $\mathscr{P}_{n,m}$ and this mapping is a bijection of $\mathscr{T}_{n,m}$ onto $\mathscr{P}_{n,m}$.

This bijection $\mathscr{T}_{n,m} \to \mathscr{P}_{n,m}$, $a \mapsto c$ can be restated as follows:

$$\forall (x, y, z) \in \mathbb{P}^3 \text{ such that } 1 \leq y + z \leq n + m:$$
$$(x, y, z) \in c \Leftrightarrow (x + n + m, y, x + z + n + m - 1) \in a. \tag{3.2}$$

Finally as a corollary of these theorems we establish a bijection between $\mathscr{B}_{n,m}$ and $\mathscr{P}_{n,m}$, Let $c = (c_{ij})_{1 \leq i \leq n+m, 1 \leq j \leq n}$ be a RCSPP in $\mathscr{P}_{n,m}$ and let k be a positive integer. Let $c_{\geq k}$ denote the plane partition formed by the parts $\geq k$. Let

$$\theta_i(c_{\geq k}) = \sharp\{l : c_{i,l} \geq k\} \tag{3.3}$$

denote the length of the ith row of $c_{\geq k}$, i.e. the rightmost column containing a letter $\geq k$ in the ith row of c.

Corollary 3.6. Let m and $n \geq 1$ be nonnegative integers and $c = (c_{ij})_{1 \leq i \leq n+m, 1 \leq j \leq n}$ be a RCSPP in $\mathscr{P}_{n,m}$. Associate to the array $c = (c_{ij})_{1 \leq i \leq n+m, 1 \leq j \leq n}$ the array $b = (b_{ij})_{1 \leq i \leq j \leq n+m-1}$ defined by

$$n - b_{ij} = \theta_{n+m-j}(c_{\geq 1-i+j}) \tag{3.4}$$

with $1 \leq i \leq j \leq n + m - 1$. Then b is in $\mathscr{B}_{n,m}$, and this mapping $\varphi_{n,m}$, which associate to a RCSPP c the TSPP $b = \varphi_{n.m}(c)$, is a bijection of $\mathscr{P}_{n,m}$ onto $\mathscr{B}_{n,m}$.

In the set-theoretic interpretation the mapping $\mathscr{P}_{n,m} \to \mathscr{B}_{n,m}$ $c \mapsto b$ is formulated by

$$\forall (x, y, z) \in \mathbb{P}^3 \text{ such that } 1 \leq x \leq y \leq n + m - 1:$$
$$(x, y, z) \in b \Leftrightarrow (n + m - y, n + 1 - z, 1 + y - x) \notin c, \tag{3.5}$$

and the mapping $\mathscr{B}_{n,m} \to \mathscr{P}_{n,m}$ $b \mapsto c$ is formulated by

$$\forall (x, y, z) \in \mathbb{P}^3 \text{ such that } 1 \leq x + z \leq n + m \text{ and } 1 \leq y \leq n:$$
$$(x, y, z) \in c \Leftrightarrow (n + m + 1 - x - z, n + m - x, n + 1 - y) \notin b. \tag{3.6}$$

Proof of Theorem 3.4. Since $\mathscr{T}_{n,m}$ is a subset of $\mathscr{T}_{n+m}$, we can imitate the proof of Theorem 1 in [26, pp.280]. Suppose that (a_{ij}) is in $\mathscr{T}_{n,m}$. Associate to the array (a_{ij}) the subarray consisting of those parts a_{ij} with $1 \leq i \leq j \leq n + m$. Then the subarray $(a_{ij})_{1 \leq i \leq j \leq n+m}$ satisfies the conditions that

(A1) all rows and columns are weakly decreasing;

(A2) $\max\{2(n+m)+1-i, n+2m\} \le a_{ij} \le 2(n+m)$ for $1 \le i \le n+m$.

Exactly the same argument as in the proof of Theorem 1 in [26, pp.280] works to show that this mapping defines a bijection from $\mathscr{T}_{n,m}$ to the set of triangular arrays $(a_{ij})_{1 \le i \le j \le n+m}$ satisfying the constraints (C1) and (C2). The reader can consult [26] to fill the details. Since $a_{ij} \ge n+2m$ and the top row of $(a_{ij})_{1 \le i \le j \le n+m}$ consists of $(n+m)$ $2(n+m)$'s, no informations is lost by omitting the top row and subtracting $(n+2m)$ from the remaining parts. When we do this, we obtain $\mathscr{B}_{n,m}$. $\square$

Proof of Theorem 3.5. Suppose that (a_{ij}) is in $\mathscr{T}_{n,m}$. Associate to the array $a = (a_{ij})_{1 \le i,j \le 2(n+m)}$ the array consisting of $a_{i+n+m,j} - (n+m)$ with $1 \le i,j \le n+m$. If $a_{i+n+m,j} - (n+m)$ is negative, then we ignore the part and regard it as zero. Then the array $\gamma = (\gamma_{ij}) = (a_{i+n+m,j} - (n+m))_{1 \le i,j \le n+m}$ satisfies the conditions that

(D1) γ is a plane partition with at most n columns;

(D2) each column of γ is a self-conjugate partition;

(D3) each part in the jth column of γ is $\le n+m-j$.

We shall show that this defines a bijection $(a_{ij}) \mapsto \gamma$ where γ satisfies the constraints (D1), (D2) and (D3). In fact γ is trivially a plane partition, and (D2) is easy to see from the symmetry. We shall show that it satisfies (D3). Suppose that $\gamma_{ij} > n+m-j$, i.e. $a_{i+n+m,j} > 2(n+m)-j$. Then we would have $(i+n+m, j, 2(n+m)-j+1) \in a$ which implies that $(n+m+1-i, j, 2(n+m)-j+1) \notin a$ by self-complementarity and total symmetry. Since $n+m+1-i < i+n+m$, this contradict the fact that a is a plane partition. Next we shall show that γ has at most n columns. In fact if γ had more than n columns then $(1, n+1, 1) \in \gamma$, i.e. $(n+m+1, n+1, n+m+1) \in a$. Since $(n+m+1, n+1, n+m+1) \in X_{2m}^{+-+}(n+m)$, this contradicts the condition (T). Thus γ satisfies (D1), (D2) and (D3), and the mapping from a to γ is well-defined.

   To see that the mapping is one-to-one we show that we can recover (a_{ij}) from γ. Let $a = (a_{ij}) \in \mathscr{T}_{n,m}$. Since a is self-complementary, all cells in $X_{2(n+m)}^{---}$ are in a, no cell in $X_{2(n+m)}^{+++}$ is in a. Since a is totally symmetric, the information in $X_{2(n+m)}^{+--}$ and $X_{2(n+m)}^{--+}$ is recovered from that of $X_{2(n+m)}^{-+-}$, and the information in $X_{2(n+m)}^{++-}$ and $X_{2(n+m)}^{-++}$ is recovered from that of $X_{2(n+m)}^{+-+}$, whereas the information in $X_{2(n+m)}^{-+-}$ is recovered from that of $X_{2(n+m)}^{+-+}$ from self-complementarity. Thus the information in $X_{2(n+m)}^{+-+}$ is exactly what we need, which is completely determined by γ.

   Finally we show that our mapping is onto. Assume that we have recovered a from γ in the above way using total symmetry and self-complementarity. We have to check a is a plane partition. For example, suppose $p = (x+n+m, y, z) \in X_{2(n+m)}^{+--}$ and $p' = (x'+n+m, y', z'+n+m) \in X_{2(n+m)}^{+-+}$ with $x \le x'$ and $y \le y'$, and $p' \in a$, i.e. $(x', y', z') \in \gamma \Rightarrow (z', y', x') \in \gamma \Rightarrow x'+y' \le n+m$ by (D2) and (D3). Thus we have $(n+m+1-x) + (n+m+1-y) \ge n+m+2$, which implies $(n+m+1-z, n+m+1-x, n+m+1-y) \notin \gamma \Rightarrow (2(n+m)+1-z, n+m+1-x, 2(n+m)+1-y) \notin a \Rightarrow p = (x+n+m, y, z) \in a$. The proofs in the other cases are similar. This shows a is a plane partition. Next we show that a satisfies

the condition (T). Since γ has at most n columns which implies $(1, n+1, 1) \notin \gamma$, i.e. $(n+m+1, n+1, n+m+1) \notin a$. From self-complementarity and total symmetry we have $(n+m, n+m, n+2m) \in a$, which implies $X_{2m}^{--+}(n+m) \subseteq a$. Similarly we have $X_{2m}^{-+-}(n+m) \subseteq a$ and $X_{2m}^{+--}(n+m) \subseteq a$, and we conclude that $X_{2m}^{-}(n+m) \subseteq a$.

Finally we use the bijection between self-conjugate partitions and strict partitions, i.e. convert each column of γ into a strict partition. Thus the map $(a_{ij})_{1 \leq i,j \leq 2(n+m)} \mapsto (\max\{a_{i+n+m,j} - (n+m) - i + 1, 0\})_{1 \leq i,j \leq n+m}$ defines a bijection of $\mathscr{T}_{n,m}$ onto $\mathscr{P}_{n,m}$. This completes the proof. $\qquad\square$

Proof of Corollary 3.6. Combining the bijections in Theorem 3.4 and Theorem 3.5, we have a bijection between $\mathscr{B}_{n,m}$ and $\mathscr{P}_{n,m}$, which is directly computed using (3.1) and (3.2). Thus (3.5) and (3.6) gives a bijection between $\mathscr{B}_{n,m}$ and $\mathscr{P}_{n,m}$. Using these identities, we have $n - b_{ij} \geq k \Leftrightarrow b_{ij} \leq n - k \Leftrightarrow (i, j, n+1-k) \notin b \Leftrightarrow (n+m-j, k, 1+j-i) \in c \Leftrightarrow c_{n+m-j,k} \geq 1+j-i$ for $1 \leq i \leq j \leq n+m-1$, which implies
$$n - b_{ij} = \sharp\{k : c_{n+m-j,k} \geq 1 - i + j\}$$
for $1 \leq i \leq j \leq n+m-1$. This proves (3.4). $\qquad\square$

4 The statistics

In [26, pp.282] Mills, Robbins and Rumsey have defined the statistics U_r (see (1.1)) for $\mathscr{B}_n$ which have been conjectured to have the same distribution as the position of the 1 in the top row of an alternating sign matrix. (See [25] for detailed explanation about the alternating sign matrices). In this section we generalize this U_r as the statistics for the generalized set $\mathscr{B}_{n,m}$ (see (4.1)), and the main goal is to translate U_r into the statistics for $\mathscr{P}_{n,m}$ (see Theorem 4.1) using the bijection in Corollary 3.6. We also define new statistics V^{R} and V^{C} which is not in [26], and give a new conjecture (see Conjecture 4.2).

Throughout this paper, for $b = (b_{ij})_{1 \leq i \leq j \leq n+m-1} \in \mathscr{B}_{n,m}$, we set $b_{i,n+m} = n - i$ for all i and $b_{0,j} = n$ for all j by convention. As an extension of these statistics we define the following statistics.

For a $b = (b_{ij})_{1 \leq i \leq j \leq n+m-1}$ in $\mathscr{B}_{n,m}$ and integers $r = 1, \ldots, n+m$, let

$$U_r(b) = \sum_{t=1}^{n+m-r} (b_{t,t+r-1} - b_{t,t+r}) + \sum_{t=n+m-r+1}^{n+m-1} \chi\{b_{t,n+m-1} > n - t\}. \qquad (4.1)$$

This $U_r(b)$ agrees with (1.1) when $m = 0$. It is easy to check that each of these functions U_r can vary between 0 and $n+m-1$ as b varies over $\mathscr{B}_{n,m}$. We put $\overline{U}_r(b) = n + m - 1 - U_r(b)$. Furthermore we identify each element in $\mathscr{P}_{n,m}$ and each element in $\mathscr{B}_{n,m}$ by the bijection $\varphi_{n,m}$ defined in Corollary 3.6, and we define $U_r(c) = U_r(\varphi_{n,m}(c))$ and $\overline{U}_r(c) = \overline{U}_r(\varphi_{n,m}(c))$ for $c \in \mathscr{P}_{n,m}$. The following theorem enable us to compute $\overline{U}_r(c)$ directly.

Theorem 4.1. Let m and $n \geq 1$ be nonnegative integers and let $c \in \mathscr{P}_{n,m}$. Then $\overline{U}_r(c)$ is the number of parts equal to r plus the number of saturated parts less than r, i.e.

$$\overline{U}_r(c) = \sharp\{(i,j) : c_{ij} = r\} + \sharp\{1 \leq k < r : c_{1,n+m-k} = k\}. \qquad (4.2)$$

Especially $\overline{U}_1(c)$ is the number of 1's in c and $\overline{U}_{n+m}(c)$ is the number of saturated parts in c. It is also easy to see that $\overline{U}_{n+m-1}(c) = \overline{U}_{n+m}(c)$ since, if a part of $c \in \mathscr{P}_{n,m}$ is equal to $n + m - 1$, then it is saturated.

We also define two new statistics. For $c \in \mathscr{P}_{n,m}$, let $V^{\mathrm{R}}(c)$ denote the number of rows of c of odd length, and let $V^{\mathrm{C}}(c)$ denote the number of columns of c of odd length. For example, $\mathscr{P}_3$ consists of the following 7 elements:

$$\emptyset \quad , \quad \boxed{1} \quad , \quad \boxed{1}\,\boxed{\mathbf{1}} \quad , \quad \boxed{\mathbf{2}} \quad , \quad \boxed{\mathbf{2}}\,\boxed{1} \quad , \quad \begin{matrix}\boxed{\mathbf{2}}\\\boxed{1}\end{matrix} \;, \quad \begin{matrix}\boxed{\mathbf{2}}\,\boxed{1}\\\boxed{1}\end{matrix} \;.$$

The distribution statistics for $\overline{U}_k(c)$, $k = 1, 2, 3$, $V^{\mathrm{R}}(c)$ and $V^{\mathrm{C}}(c)$ in $\mathscr{P}_3$ are as in Table 1 where $c \in \mathscr{P}_3$ are in this order. As the reader may easily see by comparison,

$\overline{U}_1(c)$	0	1	2	0	1	1	2
$\overline{U}_2(c)$	0	0	1	2	1	1	2
$\overline{U}_3(c)$	0	0	1	2	1	1	2
$V^{\mathrm{R}}(c)$	0	1	0	1	0	2	1
$V^{\mathrm{C}}(c)$	0	1	2	1	2	0	1

Table 1: The distribution statistics table in $\mathscr{P}_3$

the distribution of all the $\overline{U}_k(c)$, $k = 1, 2, 3$, in $\mathscr{P}_3$ is independent of k, and the function $V^{\mathrm{C}}(c)$ also have the same distribution. The first fact was proved in [26] when $m = 0$, and we also see it for general $\mathscr{P}_{n,m}$ in Section 7. We also give a proof that $V^{\mathrm{C}}(c)$ have the same distribution as $\overline{U}_k(c)$ in this paper. From this fact we also can see that the number of c in $\mathscr{P}_n$ with all columns of even length is equal to A_{n-1}. In fact. as the reader can see from the above example, the two plane partitions $\emptyset$ and $\begin{matrix}\boxed{2}\\\boxed{1}\end{matrix}$ have all columns of even length, and its generating function $\sum_{c \in \mathscr{P}_3^{\mathrm{C}}} t^{\overline{U}_k(t)} = 1 + t$, $k = 1, 2, 3$, is equal to $A_2(t)$. Further one may check the case of $c \in \mathscr{P}_n$ with all rows of even length. In the above example, the three elements $\emptyset$, $\boxed{1}\,\boxed{1}$ and $\boxed{2}\,\boxed{1}$ have all rows of even length, and its generating function is $\sum_{c \in \mathscr{P}_3^{\mathrm{R}}} t^{\overline{U}_k(t)} = 1 + t + t^2$, $k = 1, 2, 3$, which coincidently equals $A_5^{\mathrm{VS}}(t)$. It is also easily checked $\sum_{c \in \mathscr{P}_n^{\mathrm{R}}} t^{\overline{U}_k(t)}$ becomes $3(1 + t + t^2)$ if $n = 4$, $3(3 + 6t + 8t^2 + 6t^3 + 3t^4)$ if $n = 5$, $26(3 + 6t + 8t^2 + 6t^3 + 3t^4)$ if $n = 6$, and so on. From this numerical experiment one may expect the following conjecture could hold:

Conjecture 4.2. Let $n \geq 1$ be a positive integer, and let $1 \leq r \leq n$. Then

$$\sum_{c \in \mathscr{P}_n^{\mathrm{R}}} t^{\overline{U}_r(c)} = \begin{cases} A_{2m+1}^{\mathrm{VS}} \cdot A_{2m+1}^{\mathrm{VS}}(t) & \text{if } n = 2m, \\ A_{2m+1}^{\mathrm{VS}} \cdot A_{2m+3}^{\mathrm{VS}}(t) & \text{if } n = 2m + 1. \end{cases}$$

would hold. Especially, if we put $t = 1$, the number of c in $\mathscr{P}_n^{\mathrm{R}}$ would be

$$\begin{cases} \left(A_{2m+1}^{\mathrm{VS}}\right)^2 & \text{if } n = 2m, \\ A_{2m+1}^{\mathrm{VS}} \cdot A_{2m+3}^{\mathrm{VS}} & \text{if } n = 2m + 1. \end{cases}$$

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Di Francesco defined another weight in [6] for NILPs (non-intersecting lattice paths) which seems to have the same distribution as the position of the 1 in the top row of an alternating sign matrix.

Proof of Theorem 4.1. Using the identity (3.4) and $b_{n+m-r,n+m} = r - m$, we have

$$\sum_{t=1}^{n+m-r} (b_{t,t+r-1} - b_{t,t+r}) = \sum_{t=1}^{n+m-r-1} \sharp\{k : c_{t,k} \geq r+1\}$$
$$- \sum_{t=1}^{n+m-r} \sharp\{k : c_{t,k} \geq r\} + n + m - r.$$

Using $c_{t,k} \leq c_{1,k} - t + 1 \leq n + m - k - t + 1$, we have $\sharp\{k : c_{n+m-r,k} \geq r+1\} = 0$. Thus the right-hand-side is equal to $n + m - r - \sum_{t=1}^{n+m-r} \sharp\{k : c_{t,k} = r\}$. Note that $c_{n+m-r+1,1} \leq r - 1$, which implies r does not appear in the $(n+m-r+1)$st row, and $\sum_{t=1}^{n+m-r} \sharp\{k : c_{t,k} = r\}$ is the number of r appearing in c. Also by (3.4), it is easy to see that $b_{t,n+m-1} > n - t$ if and only if $c_{1,t} < n + m - t$. Thus we have

$$\sum_{t=n+m-r+1}^{n+m-1} \{b_{t,n+m-1} > n - t\} = r - 1 - \sum_{t=1}^{r-1} \chi\{c_{1,n+m-t} = t\}$$

and this shows $\overline{U}_r(c) = \sum_{t=1}^{n+m-r} \sharp\{k : c_{t,k} = r\} + \sum_{t=1}^{r-1} \chi\{c_{1,n+m-t} = t\}$. □

5 The monotone triangles and TSSCPPs

In [26, pp.287] a subset $\mathscr{B}_n^k$ of $\mathscr{B}_n$ is defined. In this section we show that we can generalize this definition naturally to the subset $\mathscr{B}_{n,m}^k$ of $\mathscr{B}_{n,m}$, and also show that there is a very nice interpretation in the words of $\mathscr{P}_{n,m}$ through the bijection defined in Section 3 (see Theorem 5.1). As a result we can present a new conjecture (see Conjecture 5.2) which is a refined version of Conjecture 1.5 ([26, Conjecture 7]). We also try to restate [26, Conjecture 7'] by means of $\mathscr{P}_{n,m}$ (see Theorem 5.4).

For $k = 0, \ldots, n+m-1$, let $\mathscr{B}_{n,m}^k$ denote the subset of those $b = (b_{ij})_{1 \leq i \leq j \leq n+m-1}$ in $\mathscr{B}_{n,m}$ such that all b_{ij} in the first $n+m-1-k$ columns are equal to their maximum values n. Also, for $k = 0, \ldots, n+m-1$, let $\mathscr{P}_{n,m}^k$ denote the subset of those $c = (c_{ij})$ in $\mathscr{P}_{n,m}$ which has at most k rows. For example, if $n = 3$ and $m = 0$, $\mathscr{B}_3$ consists of the following seven elements:

$$\begin{array}{ccccccc} \boxed{\begin{smallmatrix}3&3\\3\end{smallmatrix}} & \boxed{\begin{smallmatrix}3&3\\2\end{smallmatrix}} & \boxed{\begin{smallmatrix}3&3\\1\end{smallmatrix}} & \boxed{\begin{smallmatrix}3&2\\2\end{smallmatrix}} & \boxed{\begin{smallmatrix}3&2\\1\end{smallmatrix}} & \boxed{\begin{smallmatrix}2&2\\2\end{smallmatrix}} & \boxed{\begin{smallmatrix}2&2\\1\end{smallmatrix}} \end{array}$$

whereas $\mathscr{P}_3$ consists of the following seven plane partitions.

$$\begin{array}{ccccccc} \emptyset & 1 & 1\ \mathbf{1} & 2 & 2\ 1 & \begin{smallmatrix}2\\1\end{smallmatrix} & \begin{smallmatrix}2\ 1\\1\end{smallmatrix} \end{array}$$

There are only one element, i.e. $\emptyset$, of $\mathscr{P}_3$ with no row, five elements of $\mathscr{P}_3$ with with at most one row, and seven elements of $\mathscr{P}_3$ with at most two rows.

Theorem 5.1. Let m and $n \geq 1$ be nonnegative integers. Let $0 \leq k \leq n+m-1$. By the bijection $\varphi_{n,m}$ defined in Corollary 3.6, the subset $\mathscr{B}_{n,m}^k$ of $\mathscr{B}_{n,m}$ is in one-to-one correspondence with the subset $\mathscr{P}_{n,m}^k$ of $\mathscr{P}_{n,m}$. Especially, we have

$$\sum_{b \in \mathscr{B}_{n,m}^k} t^{U_r(b)} = \sum_{c \in \mathscr{P}_{n,m}^k} t^{\overline{U}_r(c)}.$$

We will prove Theorem 5.4 which is a refined version of this proposition and immediately implies this theorem.

By Theorem 5.1, we can reduce Conjecture 1.5 ([26, pp.287, Conjecture 7]) to the enumeration problem of $\mathscr{P}_n^k$. Furthermore, we can present the following new conjecture (Conjecture 5.2) which is a refined version of Conjecture 1.5. We give the weight to an element $m = (m_{ij})_{1 \leq i \leq j \leq n}$ of $\mathscr{M}_n^k$ by the vale $m_{n,n} - 1$. Set $M_n^k(t)$ to be the polynomial

$$\sum_{m=(m_{ij}) \in \mathscr{M}_n^k} t^{m_{n,n}-1}. \tag{5.1}$$

For instance, from the above example, one easily sees $M_3^0(t) = 1$, $M_3^1(t) = t^2 + 2t + 2$, and $M_3^2(t) = 2t^2 + 3t + 2$. The reader who is familiar with the alternating sign matrices may notice that this weight corresponds to the position of the 1 in the top row of an alternating sign matrix.

Conjecture 5.2. Let $n \geq 1$ and $1 \leq r \leq n$. Then, for $k = 0, 1, \ldots, n-1$, we would have

$$\sum_{c \in \mathscr{P}_n^k} t^{\overline{U}_r(c)} = M_n^k(t). \tag{5.2}$$

Later we will see that the left-hand side does not depend on r.

More generally the following set theoretic partitions of $\mathscr{M}_n^k$ and $\mathscr{B}_n^k$ are defined in [26]. Suppose $n \geq 1$, $0 \leq k \leq n-1$ and that x and y are nonnegative integers. Let $\mathscr{M}_n^{k,x,y}$ denote the subset of all those $m = (m_{ij})_{1 \leq i \leq j \leq n}$ in $\mathscr{M}_n^k$ such that there are precisely $x+1$ parts in column $n-k+1$ which are equal to their minimum possible value $n-k-i+2$ and $y+1$ parts in column n equal to their maximum values n. From the above example one sees that the following two elements of $\mathscr{M}_3^2$ has precisely 2 parts in the second column which are equal to their minimum possible values and exactly two parts in the last column which equal to their maximum values 3 so that $\mathscr{M}_3^{2,1,1}$ is composed of them:

$$
\begin{array}{ccc}
 & 1 & & & 2 & \\
1 & 3 & & 1 & 3 & \\
1 & 2 & 3 & 1 & 2 & 3
\end{array}
$$

A similar argument shows that $\sharp \mathscr{M}_3^{1,1,1} = \sharp \mathscr{M}_3^{1,2,0} = \sharp \mathscr{M}_3^{0,0,0} = \sharp \mathscr{M}_3^{2,0,1} = \sharp \mathscr{M}_3^{1,0,1} = \sharp \mathscr{M}_3^{1,1,0} = \sharp \mathscr{M}_3^{2,0,2} = \sharp \mathscr{M}_3^{1,0,2} = \sharp \mathscr{M}_3^{2,1,2} = 1$, $\sharp \mathscr{M}_3^{2,1,0} = \sharp \mathscr{M}_3^{2,1,1} = 2$ and the others has the cardinality zero.

Similarly, suppose $n \geq 1$, $m \geq 0$, $0 \leq k \leq n+m-1$ and that x and y are nonnegative integers. Let $\mathscr{B}_{n,m}^{k,x,y}$ be the subset of all those b in $\mathscr{B}_{n,m}^k$ such that there are precisely x entries in column $n+m-k$ which are equal to their maximum values n and there are exactly y parts $b_{i,n+m-1}$ in column $n+m-1$ equal to their minimum value $\max\{n-i, 0\}$. The following conjecture is due to Mills, Robbins and Rumsey:

Conjecture 5.3. ([26, pp.291, Conjecture 7']) The cardinality of $\mathscr{B}_n^{krs}$ is equal to the cardinality of $\mathscr{M}_n^{krs}$.

We can restate this conjecture by the following theorem. Let $\mathscr{P}_{n,m}^{k,x,y}$ be the subset of all those c in $\mathscr{P}_{n,m}^k$ such that the kth row of c has exactly $n + m - k - x$ parts and there are exactly y saturated parts in the first row of c. For example, if $n = 3$, $m = 0$, $k = 2$ and $x = y = 1$, then the following two elements of $\mathscr{P}_3^2$ has no parts in the second row and has precisely one saturated part.

$$1 \quad \mathbf{1} \qquad \mathbf{2}$$

Theorem 5.4. Let m, $n \geq 1$, x and y be nonnegative integers. Let $0 \leq k \leq n + m - 1$. By the bijection $\varphi_{n,m}$ defined in Corollary 3.6, the subset $\mathscr{B}_{n,m}^{k,x,y}$ of $\mathscr{B}_{n,m}$ is in one-to-one correspondence with the subset $\mathscr{P}_{n,m}^{k,m,n}$ of $\mathscr{P}_{n,m}$.

6 Refined strange enumeration

In [8], T. Eisenkölbl gave the (-1)-enumeration of $\mathscr{T}_n$. In [33], J. Stembridge proposed more general "strange enumeration" of CSPPs (cyclically symmetric plane partitions). The aim of this section is to restate the "strange enumeration" of $\mathscr{T}_{n,m}$ in the words of $\mathscr{P}_{n,m}$ (see Theorem 6.1) to obtain the generating functions in Section 7.

In [22, pp.25/26] natural (-1)-enumeration for the six symmetry classes of plane partitions which involve complementation are proposed (also see [8]). Here we restrict our attention to the TSSCPP case, i.e. $\mathscr{T}_{n,m}$. In this case the symmetry group G is generated by S_3 acting on the coordinates and the complementation σ_{n+m}. Then each orbit of a TSSCPP under the group action of G is always half-filled. Thus there is a natural move between TSSCPPs which replaces half of an orbit of cubes by the opposite half. Any two TSSCPPs differs by either an odd or an even number of moves, and we can define a relative sign between them. This sign becomes absolute if we assign the weight 1 to the $X_{n+m}^- \in \mathscr{T}_{n,m}$. We call this type of signed enumeration the (-1)-*enumeration* of TSSCPPs.

In the strange enumeration [33], Stembridge has proposed a way of signed enumerations of CSPPs assigning each orbit ± 1 depending on which family it belongs where each family corresponds to the faces of Coxeter complex of type A_2 modulo the action of C_3. Here we show that we can define a similar strange enumerations for TSSCPPs. Let $\langle (i, j, k) \rangle$ denote the G-orbit of a cube (i, j, k). Let

$$X_{2(n+m)}/G = P_1 \cup P_2 \cup P_3$$

denote the partition of the G-orbits of $X_{2(n+m)}$ into the families where $P_1 = \{\langle (i, i, i) \rangle\}$, $P_2 = \{\langle (i, j, j) \rangle : i \neq j\}$ and $P_3 = \{\langle (i, j, k) \rangle : i, j \text{ and } k \text{ are all distinct}\}$. For $a \in \mathscr{T}_{n,m}$, let $m_i(a)$ denote the the number of moves in P_i which is needed to obtain a from $X_{2(n+m)}^-$. Choose an arbitrary weight function $w : \mathscr{T}_{n,m} \to R$ which assigns values in some commutative ring R to each TSSCPP $a \in \mathscr{T}_{n,m}$. We may consider the signed enumeration

$$G_{n,m}(w)_{s,t,u} = \sum_{a \in \mathscr{T}_{n,m}} s^{m_1(a)} t^{m_2(a)} u^{m_3(a)} w(a)$$

where s is the sign of moves in the orbits $\langle (i, i, i) \rangle$, t is the sign of moves in the orbits $\langle (i, j, j) \rangle$ with $i \neq j$ and u is the sign of moves in the orbits $\langle (i, j, k) \rangle$ with i, j and k all distinct. Here s, t and u are ± 1, $G_{n,m}(w)_{1,1,1}$ is the ordinary generating function, and $G_{n,m}(w)_{-1,-1,-1}$ is the (-1)-enumeration. But note that s has no meaning since we need no move in P_1 to obtain a TSSCPP from $X_{2(n+m)}^-$, which implies $G_{n,m}(w)_{s,t,u}$ does not depend on s, i.e. $G_{n,m}(w)_{-1,t,u} = G_{n,m}(w)_{1,t,u}$. Thus we may consider two other strange enumerations, i.e. $G_{n,m}(w)_{1,-1,1}$ and $G_{n,m}(w)_{1,1,-1}$.

Theorem 6.1. Let m and $n \geq 1$ be non-negative integers. Let a be a TSSCPP in $\mathscr{T}_{n,m}$. Suppose a is mapped to a RCSPP c in $\mathscr{P}_{n,m}$ by the bijection defined in Theorem 3.5. Then $m_1(a) = 0$, $m_2(a)$ is the sum of parts in the first row of c and $m_3(a)$ is the sum of parts of c which is not in the first row. Especially the total number $m_1(a) + m_2(a) + m_3(a)$ of moves equals $|c|$.

In [8] Eisenkölbl considered (-1)-enumeration of several classes of plane partitions with complementary symmetry and obtain the result that the (-1)-enumeration of totally symmetric self-complementary plane partitions contained in X_{2n} equals
$$\begin{cases} A_{n+2}^{\mathrm{VS}} & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$ (see [8, Theorem 5]). By Theorem 6.1, we can consider the refined (-1)-enumeration $\sum_{c \in \mathscr{P}_{n,m}} (-1)^{|c|} t^{\overline{U}_r(c)}$ $(1 \leq r \leq n)$ and the doubly refined (-1)-enumeration $\sum_{c \in \mathscr{P}_{n,m}} (-1)^{|c|} t^{\overline{U}_1(c)} u^{\overline{U}_r(c)}$ $(2 \leq r \leq n)$ of totally symmetric self-complementary plane partitions. For c in $\mathscr{P}_{n,m}$, let $\mathrm{pr}(c)$ denote the sum of parts in the first row of c. Moreover we can propose the problem to consider the refined strange enumeration $\sum_{c \in \mathscr{P}_{n,m}} (-1)^{\mathrm{pr}(c)} t^{\overline{U}_r(c)}$ $(1 \leq r \leq n)$ and $\sum_{c \in \mathscr{P}_{n,m}} (-1)^{\mathrm{pr}(c)} (-1)^{|c|} t^{\overline{U}_r(c)}$ $(1 \leq r \leq n)$, and the doubly refined strange enumeration $\sum_{c \in \mathscr{P}_{n,m}} (-1)^{\mathrm{pr}(c)} t^{\overline{U}_1(c)} u^{\overline{U}_r(c)}$ $(2 \leq r \leq n)$ and $\sum_{c \in \mathscr{P}_{n,m}} (-1)^{\mathrm{pr}(c)} (-1)^{|c|} t^{\overline{U}_1(c)} u^{\overline{U}_r(c)}$ $(2 \leq r \leq n)$. Even the strange enumeration $\sum_{c \in \mathscr{P}_{n,m}} (-1)^{\mathrm{pr}(c)}$ and $\sum_{c \in \mathscr{P}_{n,m}} (-1)^{\mathrm{pr}(c)} (-1)^{|c|}$ (i.e. $t = u = 1$) are considered by nobody.

7 The generating functions

Using the bijection (3.4) in Corollary 3.6, we see

$$\sharp \mathscr{P}_{n,m} = \prod_{k=0}^{n-1} \frac{(3k + 3m + 1)! \prod_{i=0}^{m} (k + 2i)!}{(2k + m)!(2k + 3m + 1)! \prod_{i=1}^{m} (k + 2i - 1)!} \tag{7.1}$$

from Krattenthaler's result (see [18, Theorem 2]). In this section we give the generating function of $\mathscr{P}_{n,m}$, $\mathscr{P}_{n,m}^{\mathrm{C}}$ and $\mathscr{P}_{n,m}^{\mathrm{R}}$ with the weight $\overline{U}_k$ defined in (4.2), V^{C} and/or V^{R}, which give several refinements of of (7.1) and Pfaffian expressions for Conjecture 1.1, Conjecture 1.3, Conjecture 1.5 and Conjecture 4.2. First we give a general theorem (see Theorem 7.2) and derive the Pfaffian expression for each conjecture as a corollary of the general theorem. The reader should see Corollary 7.5 for Conjecture 1.1 and Conjecture 4.2, Corollary 7.3 for Conjecture 1.3, Corollary 7.8 for Conjecture 1.5 and Corollary 7.10 for (-1)-enumeration of $\mathscr{P}_{n,m}$. We also give two new conjectures, i.e. Conjecture 7.7 and (7.21) (7.22) in this section.

First of all we fix the notation. The reader can consult Macdonald's book [24] for information on the symmetric functions. For a positive integer l, let $\boldsymbol{x} = (x_1, x_2, \ldots, x_l)$ be an l-tuple of variables. We write the rth elementary symmetric function in $x_1, \ldots, x_l$ as $e_r^{(l)}(\boldsymbol{x}) = e_r^{(l)}(x_1, \ldots, x_l)$, i.e.

$$\sum_{r=0}^{\infty} e_r^{(l)}(\boldsymbol{x}) y^r = \prod_{i=1}^{l} (1 + x_i y).$$

Let m and $n \geq 1$ be non-negative integers. Let $\boldsymbol{x} = (x_1, \ldots, x_{n+m})$ and $\boldsymbol{t} = (t_1, \ldots, t_{n+m})$ be commutative variables. We write T_i for $\prod_{k=i}^{n+m} t_k$. For each plane partition c in $\mathscr{P}_{n,m}$, we assign the weight

$$\boldsymbol{t}^{\overline{U}(c)} \boldsymbol{x}^c = t_1^{\overline{U}_1(c)} t_2^{\overline{U}_2(c)} \cdots t_{n+m}^{\overline{U}_{n+m}(c)} x_1^{\mu_1} x_2^{\mu_2} \cdots x_{n+m}^{\mu_{n+m}},$$

where μ_i is the number of i's in c for $i = 1, \ldots, n + m$. Notice that i appears at most $n + m - i$ times in c. As an application of this Gessel-Viennot formula [9], we obtain the following fundamental lemma to enumerate the elements of $\mathscr{P}_{n,m}$:

Lemma 7.1. Let m and $n \geq 1$ be non-negative integers, and put $N = n + m$. Let λ be a partition with $\ell(\lambda) \leq n$. Then the generating function of all plane partitions $c \in \mathscr{P}_{n,m}$ of shape λ' with the weight $\boldsymbol{t}^{\overline{U}(c)} \boldsymbol{x}^c$ is given by

$$\sum_{\substack{c \in \mathscr{P}_{n,m} \\ \mathrm{sh}(c) = \lambda'}} \boldsymbol{t}^{\overline{U}(c)} \boldsymbol{x}^c = \det \left(e_{\lambda_j - j + i}^{(N-i)} (t_1 x_1, \ldots, t_{N-i-1} x_{N-i-1}, T_{N-i} x_{N-i}) \right)_{1 \leq i, j \leq n}, \quad (7.2)$$

where $T_i = \prod_{k=i}^{N} t_k$.

As a consequence of Lemma 7.1 and Theorem 2.1, we obtain the following fundamental theorem from which we can derive all the results as corollaries.

Theorem 7.2. Let m and $n \geq 1$ be non-negative integers. Let N be an even integer such that $N \geq n + m - 1$. Let A be an $(n + N) \times (n + N)$ skew-symmetric matrix. Let $B_{n,m}^N(\boldsymbol{t}, \boldsymbol{x}) = (b_{ij}^{(m)}(\boldsymbol{t}, \boldsymbol{x}))_{0 \leq i \leq n-1, \, 0 \leq j \leq n+N-1}$ be the rectangular matrix whose (i, j)th entry is

$$b_{ij}^{(m)}(\boldsymbol{t}, \boldsymbol{x}) = e_{j-i}^{(i+m)}(t_1 x_1, \ldots, t_{i+m-1} x_{i+m-1}, T_{i+m} x_{i+m}), \quad (7.3)$$

where $T_i = \prod_{k=i}^{n+m} t_k$. Then the generating function for all plane partitions $c \in \mathscr{P}_{n,m}$ with the weight $(-1)^{|\mathrm{sh}(c)|} \mathrm{Pf} \left(A \frac{\overline{I_n(\mathrm{sh}(c)')}}{I_n(\mathrm{sh}(c)')} \right) \boldsymbol{t}^{\overline{U}(c)} \boldsymbol{x}^c$ is given by

$$\sum_{\substack{\lambda \\ \ell(\lambda) \leq n}} \sum_{\substack{c \in \mathscr{P}_{n,m} \\ \mathrm{sh}(c) = \lambda'}} (-1)^{|\lambda|} \mathrm{Pf} \left(A \frac{\overline{I_n(\lambda)}}{I_n(\lambda)} \right) \boldsymbol{t}^{\overline{U}(c)} \boldsymbol{x}^c = \mathrm{Pf} \begin{pmatrix} O_n & J_n B_{n,m}^N(\boldsymbol{t}, \boldsymbol{x}) \\ -{}^t B_{n,m}^N(\boldsymbol{t}, \boldsymbol{x}) J_n & A \end{pmatrix}. \quad (7.4)$$

Here $\overline{I} = [n + N] \setminus I$ stands for the complement in the set $[n + N]$.

To specialize (7.4), we use the following notation. Let n and N be positive integers, and let m be a nonnegative integer. Let $B_{n,m}^N(t, u) = (b_{ij}^{(m)}(t, u))_{0 \leq i \leq n-1, \, 0 \leq j \leq n+N-1}$ be the $n \times (n + N)$ matrix whose (i, j)th entry is

$$b_{ij}^{(m)}(t, u) = \begin{cases} \delta_{0,j} & \text{if } i + m = 0, \\ \binom{i+m-1}{j-i} + \binom{i+m-1}{j-i-1} tu & \text{if } i + m = 1, \\ \binom{i+m-2}{j-i} + \binom{i+m-2}{j-i-1}(t + u) + \binom{i+m-2}{j-i-2} tu & \text{otherwise.} \end{cases} \quad (7.5)$$

For example,
$$B_{3,0}^2(t, u) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & tu & 0 & 0 \\ 0 & 0 & 1 & t+u & tu \end{pmatrix}.$$

We define the $n \times (n + N)$ matrices $B_{n,m}^N(t) = B_{n,m}^N(t, 1)$ and $B_{n,m}^N = B_{n,m}^N(1)$. Then the (i, j)th entry of $B_{n,m}^N(t)$ is

$$b_{ij}^{(m)}(t) = \begin{cases} \delta_{0,j} & \text{if } i + m = 0, \\ \binom{i+m-1}{j-i} + \binom{i+m-1}{j-i-1}t & \text{otherwise.} \end{cases} \qquad (7.6)$$

and the (i, j)th entry of $B_{n,m}^N$ is $\binom{i+m}{j-i}$ where the row index runs $0 \leq i \leq n-1$ and the column index runs $0 \leq j \leq n + N - 1$. When $m = 0$, these $B_{n,m}^N(t, u)$, $B_{n,m}^N(t)$ and $B_{n,m}^N$ agree with $B_n^N(t, u)$, $B_n^N(t)$ and B_n^N introduced in Section 1. The following corollary (i) gives a Pfaffian expression for the doubly refined TSSCPP conjecture (Conjecture 1.3).

Corollary 7.3. Let m and $n \geq 1$ be non-negative integers, and let N be an even integer such that $N \geq n + m - 1$.

(i) If r is a positive integer such that $2 \leq r \leq n+m$, then the generating function for all plane partitions $c \in \mathscr{P}_{n,m}$ with the weight $t^{\overline{U}_1(c)} u^{\overline{U}_r(c)}$ is

$$\sum_{c \in \mathscr{P}_{n,m}} t^{\overline{U}_1(c)} u^{\overline{U}_r(c)} = \mathrm{Pf}\begin{pmatrix} O_n & J_n B_{n,m}^N(t, u) \\ -{}^t B_{n,m}^N(t, u) J_n & \bar{S}_{n+N} \end{pmatrix}. \qquad (7.7)$$

(ii) If r is a positive integer such that $2 \leq r \leq n+m$, then the generating function for all plane partitions $c \in \mathscr{P}_{n,m}^{\mathrm{C}}$ with the weight $t^{\overline{U}_1(c)} u^{\overline{U}_r(c)}$ is

$$\sum_{c \in \mathscr{P}_{n,m}^{\mathrm{C}}} t^{\overline{U}_1(c)} u^{\overline{U}_r(c)} = \mathrm{Pf}\begin{pmatrix} O_n & J_n B_{n,m}^N(t, u) \\ -{}^t B_{n,m}^N(t, u) J_n & \bar{R}_{n+N} \end{pmatrix}. \qquad (7.8)$$

(iii) If r is a positive integer such that $1 \leq r \leq n+m$, then the generating function for all plane partitions $c \in \mathscr{P}_{n,m}$ with the weight $t^{\overline{U}_r(c)} u^{V^{\mathrm{C}}(c)}$ is

$$\sum_{c \in \mathscr{P}_{n,m}} t^{\overline{U}_r(c)} u^{V^{\mathrm{C}}(c)} = \mathrm{Pf}\begin{pmatrix} O_n & J_n B_{n,m}^N(t) \\ -{}^t B_{n,m}^N(t) J_n & \bar{C}_{n+N}(u) \end{pmatrix}. \qquad (7.9)$$

From Corollary 7.3(i)(ii) we obtain the following corollary:

Corollary 7.4. Let m and $n \geq 1$ be non-negative integers. Let r and s be integers such that $2 \leq r, s \leq n$, and let k be an integer such that $1 \leq k \leq n$. Then we have

$$\sum_{c \in \mathscr{P}_{n,m}} t^{\overline{U}_1(c)} u^{\overline{U}_r(c)} = \sum_{c \in \mathscr{P}_{n,m+1}^{\mathrm{C}}} t^{\overline{U}_1(c)} u^{\overline{U}_s(c)} = \sum_{c \in \mathscr{P}_{n,m}} t^{\overline{U}_k(c)} u^{V^{\mathrm{C}}(c)}$$

If we put $u = 1$ in Corollary 7.3(i), then we immediately obtain the following (i), which gives a progress to Conjecture 1.1. Also the following (iii) gives a Pfaffian expression for Conjecture 4.2.

Corollary 7.5. Let m and $n \geq 1$ be non-negative integers, and let N be an even integer such that $N \geq n + m - 1$.

(i) If r is a positive integer such that $1 \leq r \leq n+m$, then the generating function for all plane partitions $c \in \mathscr{P}_{n,m}$ with the weight $t^{\overline{U}_r(c)}$ is given by

$$\sum_{c \in \mathscr{P}_{n,m}} t^{\overline{U}_r(c)} = \text{Pf} \begin{pmatrix} O_n & J_n B_{n,m}^N(t) \\ -{}^t B_{n,m}^N(t) J_n & \bar{S}_{n+N} \end{pmatrix}. \tag{7.10}$$

(ii) If r is a positive integer such that $1 \leq r \leq n+m$, then the generating function for all plane partitions $c \in \mathscr{P}_{n,m}^{\text{C}}$ with all columns of even length with the weight $t^{\overline{U}_r(c)}$ is given by

$$\sum_{c \in \mathscr{P}_{n,m}^{\text{C}}} t^{\overline{U}_r(c)} = \text{Pf} \begin{pmatrix} O_n & J_n B_{n,m}^N(t) \\ -{}^t B_{n,m}^N(t) J_n & \bar{R}_{n+N} \end{pmatrix}. \tag{7.11}$$

(iii) If r is a positive integer such that $1 \leq r \leq n+m$, then the generating function for $c \in \mathscr{P}_{n,m}^{\text{R}}$ with all rows of even length with the weight $t^{\overline{U}_r(c)}$ is given by

$$\sum_{c \in \mathscr{P}_{n,m}^{\text{R}}} t^{\overline{U}_r(c)} = \text{Pf} \begin{pmatrix} O_n & J_n B_{n,m}^N(t) \\ -{}^t B_{n,m}^N(t) J_n & \bar{C}_{n+N} \end{pmatrix}. \tag{7.12}$$

(iv) The generating function for all plane partitions $c \in \mathscr{P}_{n,m}$ with the weight $t^{V^{\text{C}}(c)}$ is given by

$$\sum_{c \in \mathscr{P}_{n,m}} t^{V^{\text{C}}(c)} = \text{Pf} \begin{pmatrix} O_n & J_n B_{n,m}^N \\ -{}^t B_{n,m}^N J_n & \bar{R}_{n+N}(t) \end{pmatrix}. \tag{7.13}$$

(v) The generating function for all plane partitions $c \in \mathscr{P}_{n,m}$ with the weight $t^{V^{\text{R}}(c)}$ is given by

$$\sum_{c \in \mathscr{P}_{n,m}} t^{V^{\text{R}}(c)} = \text{Pf} \begin{pmatrix} O_n & J_n B_{n,m}^N \\ -{}^t B_{n,m}^N J_n & \bar{C}_{n+N}(t) \end{pmatrix}. \tag{7.14}$$

Note that the right-hand sides of (7.10), (7.11) and (7.12) do not depend on r. The following corollary can be proven from (7.10), (7.11) and (7.13), or is a direct consequence of Corollary 7.4.

Corollary 7.6. Let $m \geq 0$ and $n \geq 1$ be non-negative integers. Let r and s be integers such that $1 \leq r, s \leq n$, Then we have

$$\sum_{c \in \mathscr{P}_{n,m}} t^{\overline{U}_r(c)} = \sum_{c \in \mathscr{P}_{n,m+1}^{\text{C}}} t^{\overline{U}_s(c)} = \sum_{c \in \mathscr{P}_{n,m}} t^{V^{\text{C}}(c)}.$$

In particular we have $\sharp \mathscr{P}_{n,m} = \sharp \mathscr{P}_{n,m+1}^{\text{C}}$.

Corollary 7.6 reveal the relation between (7.10), (7.11) and (7.13). These are all related to the refined TSSCPP conjecture (Conjecture 1.1) when $m = 0$.

The identity (7.12) gives a Pfaffian expression for Conjecture 4.2 when $m = 0$. It is not so easy to guess the explicit form of the polynomial $\sum_{c \in \mathscr{P}^{\mathrm{R}}_{n,m}} t^{\overline{U}_r(c)}$ even for small $m \geq 2$. But, if we put $t = 1$, then $\sharp \mathscr{P}^{\mathrm{R}}_{n,m}$ can be easily guessed as follows for small m.

Conjecture 7.7. Let $n \geq 1$, $r = 0, 1$ and m be nonnegative integers. Let $f(n,m)$ denote

$$\frac{\left(6n + 6\left\lfloor \frac{m}{2} \right\rfloor + 4\right)! \left(6n + 6\left\lceil \frac{m}{2} \right\rceil + 4\right)!(2n+1)! \left(2n + 2\left\lceil \frac{m}{2} \right\rceil\right)!(2n+2m+1)! \left(n + \left\lfloor \frac{m}{2} + 1 \right\rfloor\right)!}{(4n+m+1)!(4n+m+3)!(4n+3m+2)!(4n+3m+4)! \left(2n + 2\left\lceil \frac{m}{2} \right\rceil + 1\right)! \left(n + \left\lfloor \frac{m}{2} \right\rfloor\right)!}.$$

Then the number of elements c in $\mathscr{P}^{\mathrm{R}}_{2n+r,m}$ would be

$$2^{-n} \frac{g(n, m+r)}{g(0, m+r)} \prod_{k=0}^{n-1} f(k, m+r) \tag{7.15}$$

where

$$g(n,m) = \begin{cases} h_m(n) & \text{if } \mathrm{rem}(m,4) = 0 \text{ or } 1, \\ (4n+2m+1)h_m(n) & \text{if } \mathrm{rem}(m,4) = 2 \text{ or } 3, \end{cases}$$

and $h_m(n)$ is a polynomial of degree $2\left\lfloor \frac{m}{4} \right\rfloor$ in the variable n.

For small m, $h_0(n) = h_1(n) = h_2(n) = h_3(n) = 1$, $h_4(n) = 26n^2 + 117n + 132$, $h_5(n) = 94n^2 + 517n + 715$, $h_6(n) = 526n^2 + 3419n + 5610$, $h_7(n) = 2062n^2 + 15465n + 29393$, $h_8(n) = 18788n^4 + 319396n^3 + 2042275n^2 + 5821157n + 6240360$, $h_9(n) = 8564n^4 + 162716n^3 + 1163679n^2 + 3712391n + 4457400$, and so on. The author checked this conjecture for $0 \leq m \leq 20$. For example, if $m = 6$ and $r = 1$, then the number of c in $\mathscr{P}^{\mathrm{R}}_{2n+1,6}$ would be equal to

$$2^{-n} \prod_{k=0}^{n-1} \frac{(6k+22)!(6k+28)!(2k+1)!(2k+8)!(2k+15)!(k+4)!}{(4k+8)!(4k+10)!(4k+23)!(4k+25)!(2k+9)!(k+3)!}$$
$$\times \frac{(4n+15)(2062n^2 + 15465n + 29393)}{15 \cdot 29393}$$

and the first few terms are $\sharp \mathscr{P}^{\mathrm{R}}_{3,6} = 3432$, $\sharp \mathscr{P}^{\mathrm{R}}_{5,6} = 65934024$ and $\sharp \mathscr{P}^{\mathrm{R}}_{7,6} = 9034911255456$.

Concerning (7.14), let $p^{V\mathrm{R}}_{n,m}(t) = \sum_{c \in \mathscr{P}_{n,m}} t^{V\mathrm{R}(c)}$. At this point we can guess the explicit form of the polynomial $p^{V\mathrm{R}}_{n,m}(t)$ even for $m = 0$. Let us denote $p^{V\mathrm{R}}_{n}(t) = p^{V\mathrm{R}}_{n,0}(t)$. Then we observe $p^{V\mathrm{R}}_{1}(t) = 1$, $p^{V\mathrm{R}}_{2}(t) = 1 + t$, $p^{V\mathrm{R}}_{3}(t) = t^2 + 3t + 3$, $p^{V\mathrm{R}}_{4}(t) = 3(t+1)p^{V\mathrm{R}}_{3}(t)$, $p^{V\mathrm{R}}_{5}(t) = 3(3t^4 + 18t^3 + 44t^2 + 52t + 26)$, $p^{V\mathrm{R}}_{6}(t) = 26(t+1)p^{V\mathrm{R}}_{5}(t)$ and so on.

Theorem 5.1 also give us a Pfaffian expression of Conjecture 1.5 and Conjecture 5.2 with the help of Theorem 7.2.

Corollary 7.8. Let m and $n \geq 1$ be non-negative integers, and let N be an even integer such that $N \geq n + m - 1$. If r is a positive integer such that $1 \leq r \leq n + m$, then the generating function for all plane partitions $c \in \mathscr{P}_{n,m}$ with the weight $t^{\overline{U}_r(c)}$ is given by

$$\sum_{c \in \mathscr{P}_{n,m}^k} t^{\overline{U}_r(c)} = \lim_{\varepsilon \to 0} \varepsilon^{-\lfloor \frac{k}{2} \rfloor} \mathrm{Pf} \begin{pmatrix} O_n & J_n B_{n,m}^N(t) \\ -{}^t B_{n,m}^N(t) J_n & \bar{L}_{n+N}^{(n,k)}(\varepsilon) \end{pmatrix}. \tag{7.16}$$

Especially, when $t = 1$, the number of elements of $\mathscr{P}_{n,m}^k$ is equal to

$$\lim_{\varepsilon \to 0} \varepsilon^{-\lfloor \frac{k}{2} \rfloor} \mathrm{Pf} \begin{pmatrix} O_n & J_n B_{n,m}^N \\ -{}^t B_{n,m}^N J_n & \bar{L}_{n+N}^{(n,k)}(\varepsilon) \end{pmatrix}. \tag{7.17}$$

Note that this corollary shows that $\sum_{c \in \mathscr{P}_{n,m}^k} t^{\overline{U}_r(c)}$ does not depend on r. For example, if $n = 3$, $m = 0$ and $k = 1$, then the Pfaffian in the right-hand side of (7.16) is

$$\mathrm{Pf} \left(\begin{array}{ccc|ccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 1+t & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & \varepsilon & -\varepsilon & \varepsilon & -1 & 1 & -1 \\ 0 & -1 & 0 & -\varepsilon & 0 & \varepsilon & -\varepsilon & 1 & -1 & 1 \\ -1 & -t & 0 & \varepsilon & -\varepsilon & 0 & \varepsilon & -1 & 1 & -1 \\ -1-t & 0 & 0 & -\varepsilon & \varepsilon & -\varepsilon & 0 & 1 & -1 & 1 \\ -t & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 0 \end{array} \right)$$

which equals $(t^2 + 2t + 2) + (t^2 + t)\varepsilon$. This tends to $t^2 + 2t + 2$ when $\varepsilon \to 0$. The reader may notice that there is another expression for $\sum_{c \in \mathscr{P}_{n,m}^k} t^{\overline{U}_r(c)}$ which directly follows from Theorem 5.1 and Lemma 7.1.

Remark 7.9. Let n be a positive integer and let $k = 0, 1, \ldots, n-1$. Let N be an even integer such that $N \geq k$. Let $B_{n,m}^{(k),N}(t) = \left(b_{ij}^{(m,k)}(t) \right)_{0 \leq i \leq n-1, \; 0 \leq j \leq n+N-1}$ be the $n \times (n + K)$ rectangular matrix whose $(i, j)$the entry is

$$b_{ij}^{(m,k)}(t) = \begin{cases} b_{ij}^{(m)}(t) & \text{if } 0 \leq j \leq n + k - 1, \\ 0 & \text{if } j \geq n + k \end{cases}$$

where $b_{ij}^{(m,k)}(t)$ is as in (7.5). Then we have

$$\sum_{c \in \mathscr{P}_{n,m}^k} t^{\overline{U}_r(c)} = \mathrm{Pf} \begin{pmatrix} O_n & J_n B_{n,m}^{(k),N}(t) \\ -{}^t B_{n,m}^{(k),N}(t) & \bar{S}_{n+N} \end{pmatrix}. \tag{7.18}$$

This expression (7.18) looks simpler than (7.16) apparently, but the other expression (7.16) will be more useful to derive the constant term identity (8.19) in Corollary 8.4.

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Next we consider the (-1)-enumeration of TSSCPPs. In [8], it is shown that
$$\sum_{c \in \mathscr{P}_n} (-1)^{|c|} = \begin{cases} A_{n+2}^{\mathrm{VS}} & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$ For q-binomial coefficient it is well-known
that
$$\begin{bmatrix} n \\ r \end{bmatrix}_{-1} = \begin{cases} 0 & \text{if } n \text{ is even and } r \text{ is odd,} \\ \binom{\lfloor n/2 \rfloor}{\lfloor r/2 \rfloor} & \text{otherwise.} \end{cases}$$

For the general $\mathscr{P}_{n,m}$, we can express the (-1)-enumeration of $\mathscr{P}_{n,m}$ by a Pfaffian as follows.

Corollary 7.10. Let $m \geq 0$ and $n \geq 1$ be non-negative integers. Let N be an even integer such that $N \geq n + m - 1$. Then

$$\sum_{c \in \mathscr{P}_{n,m}} (-1)^{|c|} = \mathrm{Pf} \begin{pmatrix} O_n & J_n M_{n,m}^N \\ -{}^t M_{n,m}^N J_n & \bar{S}_{n+N} \end{pmatrix}, \tag{7.19}$$

where $M_{n,m}^N = (M_{i,j}^{(m)})_{0 \leq i \leq n-1, 0 \leq j \leq n+N-1}$ is the $n \times (n + N)$ matrix defined by

$$M_{ij}^{(m)} = (-1)^{\binom{j-i+1}{2}} \begin{bmatrix} m + i \\ j - i \end{bmatrix}_{-1}. \tag{7.20}$$

From this Pfaffian expression we can see that

$$\sum_{c \in \mathscr{P}_{n,1}} (-1)^{|c|} = \sum_{c \in \mathscr{P}_{n+1}} (-1)^{|c|}$$

(this identity is also trivial from the definition) and

$$\sum_{c \in \mathscr{P}_{n,3}} (-1)^{|c|} = \sum_{c \in \mathscr{P}_{n+3}} (-1)^{|c|}.$$

For $m = 2, 4$ we can observe the following conjecture:

$$\sum_{c \in \mathscr{P}_{n,2}} (-1)^{|c|} = \begin{cases} 3^k \prod_{i=0}^{k-1} \frac{(6i+4)!(3i+5)!(2i+1)!(2i+3)!(i+1)!}{(4i+3)!(4i+6)!(3i+3)!(2i)!(i+2)!} & \text{if } n = 2k \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{7.21}$$

and

$$\sum_{c \in \mathscr{P}_{n,4}} (-1)^{|c|} = \sum_{c \in \mathscr{P}_{n+2,2}} (-1)^{|c|} \tag{7.22}$$

would hold. For example, the first few terms of $\sum_{c \in \mathscr{P}_{n,2}} (-1)^{|c|}$ are 1, 4, 50, 1862, 202860, and so on. For $m \geq 5$, it seems hard to guess the explicit form of $\sum_{c \in \mathscr{P}_{n,m}} (-1)^{|c|}$.

From Theorem 6.1, $\sum_{c \in \mathscr{P}_{n,m}} (-1)^{|c|} t^{\overline{U}_r(c)}$ and $\sum_{c \in \mathscr{P}_{n,m}} (-1)^{|c|} t^{V^C(c)}$ can be regarded as refined (-1)-enumeration of TSSCPPs. Meanwhile, in this case, we should note that $\sum_{c \in \mathscr{P}_{n,m}} (-1)^{|c|} t^{\overline{U}_r(c)}$ does depend on r. Recall that $\overline{U}_1(c)$ is the number of 1's in c, and $\overline{U}_{n+m}(c)$ is the number of saturated parts in c. These two cases can be easily described as in the following corollary (i)(ii) which is obtained as a corollary of Theorem 7.2.

Corollary 7.11. Let m and $n \geq 1$ be non-negative integers, and let N be an even integer such that $N \geq n + m - 1$.

(i) We have

$$\sum_{c \in \mathscr{P}_{n,m}} (-1)^{|c|} t^{\overline{U}_1(c)} = \mathrm{Pf} \begin{pmatrix} O_n & J_n M_{n,m}^{\overline{U}_1,N}(t) \\ -{}^t M_{n,m}^{\overline{U}_1,N}(t) J_n & \bar{S}_{n+N} \end{pmatrix}, \qquad (7.23)$$

where $M_{n,m}^{\overline{U}_1,N}(t) = \left(M_{ij}^{(m),\overline{U}_1}(t) \right)_{0 \leq i \leq n-1, 0 \leq j \leq n+N-1}$ is defined by

$$M_{ij}^{(m),\overline{U}_1}(t) = (-1)^{\binom{j-i+1}{2}} \left\{ (-1)^{j-i} \begin{bmatrix} m+i-1 \\ j-i \end{bmatrix}_{-1} + \begin{bmatrix} m+i-1 \\ j-i-1 \end{bmatrix}_{-1} t \right\}.$$

(ii) We have

$$\sum_{c \in \mathscr{P}_{n,m}} (-1)^{|c|} t^{\overline{U}_{n+m}(c)} = \mathrm{Pf} \begin{pmatrix} O_n & J_n M_{n,m}^{\overline{U}_{n+m},N}(t) \\ -{}^t M_{n,m}^{\overline{U}_{n+m},N}(t) J_n & \bar{S}_{n+N} \end{pmatrix}, \qquad (7.24)$$

where $M_{n,m}^{\overline{U}_{n+m},N}(t) = \left(M_{ij}^{(m),\overline{U}_{n+m}}(t) \right)_{0 \leq i \leq n-1, 0 \leq j \leq n+N-1}$ is defined by

$$M_{ij}^{(m),\overline{U}_{n+m}}(t) = (-1)^{\binom{j-i+1}{2}} \left\{ \begin{bmatrix} m+i-1 \\ j-i \end{bmatrix}_{-1} + (-1)^{m+2i-j} \begin{bmatrix} m+i-1 \\ j-i-1 \end{bmatrix}_{-1} t \right\}.$$

(iii) We have

$$\sum_{c \in \mathscr{P}_{n,m}} (-1)^{|c|} t^{V^C(c)} = \mathrm{Pf} \begin{pmatrix} O_n & J_n M_{n,m}^N \\ -{}^t M_{n,m}^N J_n & \bar{R}_{n+N}(t) \end{pmatrix}. \qquad (7.25)$$

If one puts $m = 0$ in (7.23), (7.24) and (7.25) and checks the first few terms of $\sum_{c \in \mathscr{P}_n} (-1)^{|c|} t^{\overline{U}_1(c)}$, $\sum_{c \in \mathscr{P}_n} (-1)^{|c|} t^{\overline{U}_n(c)}$ and $\sum_{c \in \mathscr{P}_n} (-1)^{|c|} t^{V^C(c)}$, then he will see that neither of these polynomials equals $A_{n+2}^{\mathrm{VS}}(t)$ when n is odd. For example, the first few terms of $\sum_{c \in \mathscr{P}_n} (-1)^{|c|} t^{V^C(c)}$ are 1, $t-1$, t, $(t-1)(t^2-t+1)$, $t(t^2+t+1)$, $(t-1)(t^2+1)(3t^2-4t+3)$, $2t(2t^4+3t^3+3t^2+3t+2)$ and $2(t-1)(13t^6-20t^5+37t^4-35t^3+37t^2-20t+13)$. It will be an interesting problem to find a new weight whose distribution gives the polynomial $A_{n+2}^{\mathrm{VS}}(t)$.

We are now in the position to give proofs. First we recall notation and definitions used for the lattice path method due to Gessel and Viennot [9]. Let $D = (V, E)$ be an acyclic digraph without multiple edges. If u and v are any pair of vertices, let $\mathscr{P}(u, v)$ denote the set of all directed D-paths from u to v. For a fixed positive integer n, an n-vertex is an n-tuple of vertices of D. If $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ are n-vertices, an n-path from $\boldsymbol{u}$ to $\boldsymbol{v}$ is an n-tuple $\boldsymbol{P} = (P_1, \ldots, P_n)$ such that $P_i \in \mathscr{P}(u_i, v_i)$, $i = 1, \ldots, n$. The n-path $\boldsymbol{P} = (P_1, \ldots, P_n)$ is said to be *non-intersecting* if any two different paths P_i and P_j have no vertex in common. We will write $\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})$ for the set of all n-paths from $\boldsymbol{u}$ to $\boldsymbol{v}$, and write $\mathscr{P}_0(\boldsymbol{u}, \boldsymbol{v})$

for the subset of $\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})$ consisting of non-intersecting n-paths. If $\boldsymbol{u} = (u_1, \ldots, u_m)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ are linearly ordered sets of vertices of D, then $\boldsymbol{u}$ is said to be *D-compatible* with $\boldsymbol{v}$ if every path $P \in \mathscr{P}(u_i, v_l)$ intersects with every path $Q \in \mathscr{P}(u_j, v_k)$ whenever $i < j$ and $k < l$. Let S_n denote the symmetric group on $\{1, 2, \ldots, n\}$. Then for $\pi \in S_n$, by $\boldsymbol{v}^\pi$ we mean the n vertex $(v_{\pi(1)}, \ldots, v_{\pi(n)})$.

We assign a commutative indeterminate x_e to each edge e of D and call it the weight of the edge. Set the weight of a path P to be the product of the weights of its edges and denote it by $w(P)$. The weight $w(\boldsymbol{P})$ of an n-path $\boldsymbol{P}$ is defined to be the product of the weights of its components. Given any family $\mathscr{F}$ of edge multi-sets, we will write GF $[\mathscr{F}]$ for the generating function with respect to the weight function w. Thus, if $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ are n-vertices, we define the generating functions $F(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{GF}[\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})] = \sum_{\boldsymbol{P} \in \mathscr{P}(\boldsymbol{u}, \boldsymbol{v})} w(\boldsymbol{P})$ and $F_0(\boldsymbol{u}, \boldsymbol{v}) = \mathrm{GF}[\mathscr{P}_0(\boldsymbol{u}, \boldsymbol{v})] = \sum_{\boldsymbol{P} \in \mathscr{P}_0(\boldsymbol{u}, \boldsymbol{v})} w(\boldsymbol{P})$. In particular, if u and v are any pair of vertices, we write

$$h(u, v) = \mathrm{GF}[\mathscr{P}(u, v)] = \sum_{P \in \mathscr{P}(u, v)} w(P).$$

The following lemma is called the Gessel-Viennot formula for counting lattice paths in terms of determinants. (See [9, 16, 32].)

Lemma 7.12. (Lidström-Gessel-Viennot)
Let $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ be two n-vertices in an acyclic digraph D. Then

$$\sum_{\pi \in S_n} \mathrm{sgn}\pi \; F_0(\boldsymbol{u}^\pi, \boldsymbol{v}) = \det[h(u_i, v_j)]_{1 \le i, j \le n}. \tag{7.26}$$

In particular, if $\boldsymbol{u}$ is D-compatible with $\boldsymbol{v}$, then

$$F_0(\boldsymbol{u}, \boldsymbol{v}) = \det[h(u_i, v_j)]_{1 \le i, j \le n}. \tag{7.27}$$

Proof of Lemma 7.1. We give a lattice path realization of each $c \in \mathscr{P}_{n,m}$. Let $V = \{(x, y) \in \mathbb{N}^2 : 0 \le y \le x\}$ be the vertex set, and direct an edge from u to v whenever $v - u = (1, -1)$ or $(0, -1)$.

(i) We assign the weight

$$\begin{cases} \prod_{k=j}^N t_k \cdot x_j & \text{if } j = i, \\ t_j x_j & \text{if } j < i, \end{cases}$$

to the horizontal edge from $u = (i, j)$ to $v = (i + 1, j - 1)$.

(ii) We assign the weight 1 to the vertical edge from $u = (i, j)$ to $v = (i, j - 1)$.

Let $u_j = (N - j, N - j)$ and $v_j = (\lambda_j + N - j, 0)$ for $j = 1, \ldots, n$, and let $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$. We claim that the RCSPPs $c \in \mathscr{P}_{n,m}$ of shape λ' can be identified as n-tuples of nonintersecting D-paths in $\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})$. To see this, consider the correspondence between the following plane partition and the paths

illustrated in Figure 2:

8	**8**	**7**	5	**5**	3	**3**
7	7	6	3	3	2	
5	5	5	2	2		
3	2	2	1	1		
2	1	1				
1						

The jth path P_j from u_j to v_j corresponds to the jth column of c. The entries in this column can be obtained by reading the second coordinates of the horizontal steps of P_j from left to right. By (4.1), each horizontal step from (i,j) to $(i+1, j-1)$ contributes t_j to $\boldsymbol{t}^{\overline{U}(c)}$ if $j < i$, and contributes T_j to $\boldsymbol{t}^{\overline{U}(c)}$ if $j = i$, and this is realized by the above asigned weights to each edge. Note that the generating function $h(u_i, v_j)$ is an elementary symmetric function, i.e.

$$h(u_i, v_j) = e^{(N-i)}_{\lambda_j - j + i}(t_1 x_1, \ldots, t_{N-i-1} x_{N-i-1}, T_{N-i} x_{N-i}).$$

Thus we obtain the desired result (7.2) from (7.27). This completes the proof. $\square$

Now we are in the position to give a proof of Theorem 7.2.

Proof of Theorem 7.2. If $(M_{ij})_{1 \le i,j \le n}$ is any $n \times n$ matrix, we have $\det(M_{ij}) = \det(M_{n+1-i, n+1-j})$ in general. Thus the determinant in the right-hand side of (7.2) is equal to

$$\det\left(e^{(m+i)}_{\lambda_{n-j}+j-i}(t_1 x_1, \ldots, t_{m+i-1} x_{m+i-1}, T_{m+i} x_{m+i})\right)_{0 \le i,j \le n-1}.$$

Note that the column indices are $\{\lambda_{n-j} + j | 0 \le j \le n-1\} = I_n(\lambda)$, and the weight of this determinant is $\mathrm{Pf}\left(A^{I_n(\lambda)}_{I_n(\lambda)}\right)$. If we take the sum over all partitions, we obtain the desired identity (7.4) from the minor summation formula (2.9). This complete the proof. $\square$

Proof of Corollary 7.3. To prove (i), substitute $t_1 = t$, $t_k = u$, $t_i = 1$ ($i \ne 1, k$), $x_i = 1$ ($1 \le i \le n+m$), and $A = \bar{S}_{n+N}$ into (7.4). To prove (ii), substitute $t_1 = t$, $t_i = 1$ ($i = 2, \ldots, n+m$), $x_i = 1$ ($1 \le i \le n+m$), and $A = \bar{C}_{n+N}(t)$ into (7.4). $\square$

Proof of Corollary 7.4. We consider the right-hand side of (7.9) where N should be taken large enough. Here we assume the row and column indices run over $[2n+N]$.

For example, if $n = 3$, $m = 0$ and $N = 7$ then the right-hand side of (7.9) looks like

$$\text{Pf} \left(\begin{array}{ccc|ccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 1+t & t & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & t & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 0 & 1 & -u & u^2 & -u^3 & u^4 & -u^5 \\
0 & -1 & 0 & -1 & 0 & 1 & -u & u^2 & -u^3 & u^4 \\
-1 & -t & 0 & u & -1 & 0 & 1 & -u & u^2 & -u^3 \\
-1-t & 0 & 0 & -u^2 & u & -1 & 0 & 1 & -u & u^2 \\
-t & 0 & 0 & u^3 & -u^2 & u & -1 & 0 & 1 & -u \\
0 & 0 & 0 & -u^4 & u^3 & -u^2 & u & -1 & 0 & 1 \\
0 & 0 & 0 & u^5 & -u^4 & u^3 & -u^2 & u & -1 & 0
\end{array} \right).$$

Add u times column $2n + N - 1$ to column $2n + N$, add u times column $2n + N - 2$ to column $2n + N - 1$, ..., and add u times column $n + 2$ to column $n + 3$. Then, add tu times column $n+1$ to column $n+2$ if $m = 0$, or add u times column $n+1$ to column $n + 2$ otherwise. Perform the same operation on the rows. Thus we obtain the right-hand side of (7.8) where m is replaced by $m + 1$. In the above example, the resulting matrix looks like

$$\text{Pf} \left(\begin{array}{ccc|ccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 1+t+u & t+u+tu & tu & 0 \\
0 & 0 & 0 & 0 & 1 & t+u & tu & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & tu & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & -t & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
-1-t-u & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
-t-u-tu & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
-tu & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{array} \right).$$

This proves the second equality. To prove the first equality, we perform similar operations on the right-hand side of (7.7). $\qquad\square$

Proof of Corollary 7.5. To prove (i), Substitute $t_i = 1$ $(1 \le i \le n + m)$, $x_i = 1$ $(1 \le i \le n + m)$ and $A = \bar{S}_{n+N}$ into (7.4). To prove (ii), Substitute $t_i = 1$ $(1 \le i \le n + m)$, $x_i = 1$ $(1 \le i \le n + m)$ and $A = \bar{R}_{n+N}$ into (7.4) since $\text{sh}(c)'$ should be even. The other identities can be proven similarly using (7.4) and Proposition 2.3(iii). $\qquad\square$

Proof of Corollary 7.8. As before, we substitute $t_1 = t$, $t_i = 1$ $(2 \le i \le n + m)$, $x_i = 1$ $(1 \le i \le n + m)$ and $A = \bar{L}_{n+N}^{(n,k)}(\varepsilon)$ into (7.4). Proposition 2.3(iii) proves our claim. $\qquad\square$

Proof of Corollary 7.11. To prove (7.23), we substitute $x_i = q^i$ for $i = 1, \ldots, n + m$, $t_1 = t$, and $t_k = 1$ for $k = 2, \ldots, n + m$ into (7.4). Since $e_r^{(n)}(x_1, \ldots, x_n) = e_r^{(n-1)}(x_2, \ldots, x_n) + x_1 e_{r-1}^{(n-1)}(x_2, \ldots, x_n)$ and $e_r^{(n)}(q, q^2, \ldots, q^n) = q^{\binom{r+1}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_q$, we obtain

$$e_r^{(n)}(tq, q^2, \ldots, q^n) = q^{\frac{r(r+1)}{2}} \left\{ q^r \begin{bmatrix} n-1 \\ r \end{bmatrix}_q + \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q t \right\}.$$

Thus, if we put $q = -1$, then the entries $b_{ij}^{(m)}(\boldsymbol{t}, \boldsymbol{x})$ in (7.3) becomes

$$= (-1)^{\frac{(j-i)(j-i+1)}{2}} \left\{ (-1)^{j-i} \begin{bmatrix} n-1 \\ r \end{bmatrix}_{-1} + \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{-1} t \right\}.$$

If we substitute $A = \bar{S}_{n+N}$ then we obtain the desired identity from Proposition 2.3. The other two identities (7.24) and (7.25) can be shown similarly. $\square$

8   Constant term identities

In [35], D. Zeilberger proved the following constant term identity. Let D be the sum of all the $n \times n$ minors of the $n \times (2n + m - 1)$ matrix X given by

$$X_{ij} = \binom{m+i}{j-i}, \qquad 0 \le i \le n-1, \quad 0 \le j \le 2n + m - 2,$$

and let C be the constant term of

$$\prod_{1 \le i \le j \le n} \left(1 - \frac{z_i}{z_j} \right) \prod_{i=1}^{n} \left(1 + \frac{1}{z_i} \right)^{m+n-i} \prod_{i=1}^{n} \frac{1}{1 - z_i} \prod_{1 \le i \le j \le n} \frac{1}{1 - z_i z_j},$$

then $D = C$ holds. The aim of this section is to give a generalization of this constant term identity, which gives the constant term identities for all conjectures we treat. In that sense, Theorem 8.1 gives a generalization of Zeilberger's theorem, and, as corollaries, we obtain Corollary 8.2 for the doubly refined TSSCPP conjecture, Corollary 8.3 for the refined TSSCPP conjecture and Corollary 8.4 for Conjecture 1.5, which are the main results of this section.

Let m and $n \ge 1$ be non-negative integers. Let N be an even integer such that $N \ge n + m - 1$. Let A be an $(n + N) \times (n + N)$ skew-symmetric matrix. Let $B_{n,m}^N(t) = (b_{ij}^{(m)}(t))_{0 \le i \le n-1, 0 \le j \le n+N-1}$ be the $n \times (n+N)$ matrix defined in (7.5). Let $D_{n,m}(A, t, u)$ be the sum

$$\sum_{I \in \binom{[n+N]}{n}} (-1)^{s(\bar{I}, I)} \mathrm{Pf}\left(A_{\bar{I}}^{\bar{I}} \right) \det \left(B_{n,m}^N(t, u)_I \right), \tag{8.1}$$

and we also write $D_{n,m}(A, t)$ for $D_{n,m}(A, t, 1)$, and $D_{n,m}(A)$ for $D_{n,m}(A, 1)$. Let T be the $n \times (n + N)$ matrix whose (i, j)th entry is z_i^{j-1} for $1 \le i \le n$ and $1 \le j \le n+N$. If we put

$$G_A(z_1, \cdots, z_n) = \mathrm{Pf} \begin{pmatrix} O_n & J_n T \\ -{}^t T J_n & A \end{pmatrix},$$

then $G_A(z_1, \cdots, z_n)$ is an anti-symmetric polynomial in the variables $z_1, \ldots, z_n$, so that we can write $G_A(z_1, \cdots, z_n) = F_A(z_1, \cdots, z_n) \prod_{1 \le i < j \le n} (z_j - z_i)$ with a symmetric polynomial $F_A(z_1, \cdots, z_n)$. For example, if we take $\bar{S}_\infty$, $\bar{R}_\infty$ and $\bar{S}_\infty$ for A,

we can easily see that

$$F_{\bar{S}_\infty}(z_1, \cdots, z_n) = \sum_\lambda s_\lambda^{(n)}(\boldsymbol{x}) = \prod_{i=1}^n \frac{1}{1 - z_i} \prod_{1 \le i < j \le n} \frac{1}{1 - z_i z_j}, \tag{8.2}$$

$$F_{\bar{R}_\infty}(z_1, \cdots, z_n) = \sum_{\lambda \text{ even}} s_\lambda^{(n)}(\boldsymbol{x}) = \prod_{i=1}^n \frac{1}{1 - z_i^2} \prod_{1 \le i < j \le n} \frac{1}{1 - z_i z_j}, \tag{8.3}$$

$$F_{\bar{C}_\infty}(z_1, \cdots, z_n) = \sum_{\lambda' \text{ even}} s_\lambda^{(n)}(\boldsymbol{x}) = \prod_{1 \le i < j \le n} \frac{1}{1 - z_i z_j}, \tag{8.4}$$

where $s_\lambda^{(n)}(\boldsymbol{x})$ denotes the Schur function in the n variables $\boldsymbol{x} = (x_1, \dots, x_n)$ corresponding to the partition λ (see [24, I, 5, Ex.4, 5]). It can be also shown that

$$F_{\bar{R}_\infty(t)}(z_1, \cdots, z_n) = \sum_\lambda t^{r(\lambda)} s_\lambda^{(n)}(\boldsymbol{x}) = \prod_{i=1}^n \frac{1 + t z_i}{1 - z_i^2} \prod_{1 \le i < j \le n} \frac{1}{1 - z_i z_j}, \tag{8.5}$$

$$F_{\bar{C}_\infty(t)}(z_1, \cdots, z_n) = \sum_\lambda t^{c(\lambda)} s_\lambda^{(n)}(\boldsymbol{x}) = \prod_{i=1}^n \frac{1}{1 - t z_i} \prod_{1 \le i < j \le n} \frac{1}{1 - z_i z_j}, \tag{8.6}$$

(see [24, I, 5, Ex.7, 8]). To derive these identities are the original motivation of the minor summation formula (see [11, 14, 15]). I.G. Macdonald obtained the bounded version of (8.2):

$$\lim_{\varepsilon \to 0} \varepsilon^{-\lfloor \frac{n}{2} \rfloor} F_{\bar{L}_\infty^{(n,k)}(\varepsilon)}(z_1, \cdots, z_n) = \sum_{\substack{\lambda \\ \lambda_1 \le k}} s_\lambda^{(n)}(\boldsymbol{x}) = \frac{\det(z_i^{j-1} - z_i^{k+2n-j})_{1 \le i,j \le n}}{\prod_{i=1}^n (1 - z_i) \prod_{1 \le i < j \le n}(z_j - z_i)(1 - z_i z_j)}, \tag{8.7}$$

(see [24, I, 5, Ex.16]). In fact, this identity (8.7) can be also derived from the minor summation formula (2.12), (2.14) and Schur's Pfaffian [13] (also see [11]).

If we write $h_i^{(m)}(z, t, u) = \sum_{j \ge 0} b_{ij}^{(m)}(t, u) z^{j-i}$ where $b_{ij}^{(m)}(t)$ is as in (7.5), then we have

$$h_i^{(m)}(z, t, u) = \begin{cases} (1+z)^{m+i} & \text{if } m + i = 0, \\ (1+z)^{m+i-1}(1 + tuz) & \text{if } m + i = 1, \\ (1+z)^{m+i-2}(1 + tz)(1 + uz) & \text{otherwise.} \end{cases} \tag{8.8}$$

We also write $h_i^{(m)}(z, t)$ for $h_i^{(m)}(z, t, 1)$, and $h_i^{(m)}(z) = h_i^{(m)}(z, 1) = (1+z)^{m+i}$.

Theorem 8.1. Let $C_{n,m}(A, t, u)$ denote the constant term of

$$\prod_{1 \le i < j \le n} \left(1 - \frac{z_j}{z_i}\right) \prod_{k=1}^n h_{n-k}^{(m)}(z_k^{-1}, t, u) F_A(z_1, \cdots, z_n). \tag{8.9}$$

Then $D_{n,m}(A, t, u)$ is equal to $C_{n,m}(A, t, u)$.

The following corollary gives an constant term expression for the doubly refined enumeration of TSSCPPs. See Corollary 7.4 for the relations between these constant terms.

Corollary 8.2. Let m and $n \geq 1$ be non-negative integers.

(i) If r is an integer such that $2 \leq r \leq n+m$, then $\sum_{c \in \mathscr{P}_{n,m}} t^{\overline{U}_1(c)} u^{\overline{U}_r(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{z}} \prod_{1 \leq i < j \leq n} \left(1 - \frac{z_j}{z_i}\right) \prod_{k=1}^{n} h_{n-k}^{(m)}(z_k^{-1}, t, u) \prod_{i=1}^{n} \frac{1}{1 - z_i} \prod_{1 \leq i < j \leq n} \frac{1}{1 - z_i z_j}. \quad (8.10)$$

(ii) If r is an integer such that $2 \leq r \leq n+m$, then $\sum_{c \in \mathscr{P}_{n,m}^{\mathrm{C}}} t^{\overline{U}_1(c)} u^{\overline{U}_r(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{z}} \prod_{1 \leq i < j \leq n} \left(1 - \frac{z_j}{z_i}\right) \prod_{k=1}^{n} h_{n-k}^{(m)}(z_k^{-1}, t, u) \prod_{i=1}^{n} \frac{1}{1 - z_i^2} \prod_{1 \leq i < j \leq n} \frac{1}{1 - z_i z_j}. \quad (8.11)$$

(iii) If r is an integer such that $1 \leq r \leq n+m$, then $\sum_{c \in \mathscr{P}_{n,m}} t^{\overline{U}_1(c)} u^{V^{\mathrm{C}}(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{z}} \prod_{1 \leq i < j \leq n} \left(1 - \frac{z_j}{z_i}\right) \prod_{k=1}^{n} h_{n-k}^{(m)}(z_k^{-1}, t) \prod_{i=1}^{n} \frac{1}{1 - u z_i} \prod_{1 \leq i < j \leq n} \frac{1}{1 - z_i z_j}. \quad (8.12)$$

The following corollary gives an constant term expression for the refined enumeration of TSSCPPs. See Corollary 7.6 for the relations between these constant terms.

Corollary 8.3. Let m and $n \geq 1$ be non-negative integers.

(i) If r is an integer such that $1 \leq r \leq n+m$, then $\sum_{c \in \mathscr{P}_{n,m}} t^{\overline{U}_r(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{z}} \prod_{1 \leq i < j \leq n} \left(1 - \frac{z_j}{z_i}\right) \prod_{k=1}^{n} h_{n-k}^{(m)}(z_k^{-1}, t) \prod_{i=1}^{n} \frac{1}{1 - z_i} \prod_{1 \leq i < j \leq n} \frac{1}{1 - z_i z_j}. \quad (8.13)$$

(ii) If r is an integer such that $1 \leq r \leq n+m$, then $\sum_{c \in \mathscr{P}_{n,m}^{\mathrm{C}}} t^{\overline{U}_r(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{z}} \prod_{1 \leq i < j \leq n} \left(1 - \frac{z_j}{z_i}\right) \prod_{k=1}^{n} h_{n-k}^{(m)}(z_k^{-1}, t) \prod_{i=1}^{n} \frac{1}{1 - z_i^2} \prod_{1 \leq i < j \leq n} \frac{1}{1 - z_i z_j}. \quad (8.14)$$

(iii) If r is an integer such that $1 \leq r \leq n+m$, then $\sum_{c \in \mathscr{P}_{n,m}^{\mathrm{R}}} t^{\overline{U}_r(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{z}} \prod_{1 \leq i < j \leq n} \left(1 - \frac{z_j}{z_i}\right) \prod_{k=1}^{n} h_{n-k}^{(m)}(z_k^{-1}, t) \prod_{1 \leq i < j \leq n} \frac{1}{1 - z_i z_j}. \quad (8.15)$$

In particular if we put $t = 1$ in this equation, then we see that $\sharp \mathscr{P}_{n,m}^{\mathrm{R}}$ equals

$$\mathrm{CT}_{\boldsymbol{z}} \prod_{1 \leq i < j \leq n} \left(1 - \frac{z_j}{z_i}\right) \prod_{k=1}^{n} \left(1 + \frac{1}{z_k}\right)^{n+m-k} \prod_{i=1}^{n} \frac{1}{1 - z_i^2} \prod_{1 \leq i < j \leq n} \frac{1}{1 - z_i z_j}. \quad (8.16)$$

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(iv) The generating function $\sum_{c\in\mathscr{P}_{n,m}} t^{V^C(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{z}} \prod_{1\leq i<j\leq n} \left(1 - \frac{z_j}{z_i}\right) \prod_{k=1}^n \left(1 + \frac{1}{z_k}\right)^{n+m-k} \prod_{i=1}^n \frac{1}{1 - tz_i} \prod_{1\leq i<j\leq n} \frac{1}{1 - z_iz_j}. \quad (8.17)$$

(v) The generating function $\sum_{c\in\mathscr{P}_{n,m}} t^{V^R(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{z}} \prod_{1\leq i<j\leq n} \left(1 - \frac{z_j}{z_i}\right) \prod_{k=1}^n \left(1 + \frac{1}{z_k}\right)^{n+m-k} \prod_{i=1}^n \frac{1 + tz_i}{1 - z_i^2} \prod_{1\leq i<j\leq n} \frac{1}{1 - z_iz_j}. \quad (8.18)$$

The following corollary gives a constant term identity to answer Conjecture 5.2 and Conjecture 1.5.

Corollary 8.4. Let m and $n \geq 1$ be non-negative integers. If r is an integer such that $1 \leq r \leq n + m$, then $\sum_{c\in\mathscr{P}_{n,m}^k} t^{\overline{U}_r(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{z}} \prod_{1\leq i<j\leq n} \left(1 - \frac{z_j}{z_i}\right) \prod_{i=1}^n h_{n-i}^{(m)}(z_i^{-1}, t) \frac{\det(z_i^{j-1} - z_i^{k+2n-j})_{1\leq i,j\leq n}}{\prod_{i=1}^n (1 - z_i) \prod_{1\leq i<j\leq n}(z_j - z_i)(1 - z_iz_j)}. \quad (8.19)$$

Especially, when $t = 1$, the number of elements of $\mathscr{P}_{n,m}^k$ is equal to

$$\mathrm{CT}_{\boldsymbol{z}} \prod_{1\leq i<j\leq n} \left(1 - \frac{z_j}{z_i}\right) \prod_{i=1}^n \left(1 + \frac{1}{z_i}\right)^{n+m-k} \frac{\det(z_i^{j-1} - z_i^{k+2n-j})_{1\leq i,j\leq n}}{\prod_{i=1}^n (1 - z_i) \prod_{1\leq i<j\leq n}(z_j - z_i)(1 - z_iz_j)}. \quad (8.20)$$

Christian Krattenthaler has obtained an equivalent result to (8.20) in [20] concerning Conjecture 1.5 (i.e. Conjecture 7 of [26]).

Proof of Theorem 8.1. We use the notation $\boldsymbol{z} = (z_1, \ldots, z_n)$ and let $\mathrm{CT}_{\boldsymbol{z}}$ denote the constant term in $\boldsymbol{z}$. We also write $\alpha(I) = (-1)^{s(\overline{I},I)}\mathrm{Pf}\left(A_{\overline{I}}^{\overline{I}}\right)$ for brevity. Then $D_{n,m}(A, t, u)$ is equal to

$$\sum_{I=\{j_1,\ldots,j_n\}\in\binom{[n+N]}{n}} \alpha(I) \det\left(b_{i-1,j_k-1}^{(m)}(t, u)\right)_{1\leq i,k\leq n}$$

$$= \sum_{I=\{j_1,\ldots,j_n\}\in\binom{[n+N]}{n}} \alpha(I) \sum_{\pi\in S_n} \mathrm{sgn}\,\pi \prod_{k=1}^n b_{\pi(k)-1,j_k-1}^{(m)}(t, u).$$

This sum equals

$$\sum_{I=\{j_1,\ldots,j_n\}\in\binom{[n+N]}{n}} \alpha(I)\mathrm{CT}_{\boldsymbol{z}} \sum_{\pi\in S_n} \mathrm{sgn}\,\pi \prod_{k=1}^n \frac{h_{\pi(k)-1}^{(m)}(z_{\pi(k)}, t, u)}{z_k^{j_k-\pi(k)}}$$

$$= \sum_{I=\{j_1,\ldots,j_n\}\in\binom{[n+N]}{n}} \alpha(I)\mathrm{CT}_{\boldsymbol{z}} \prod_{k=1}^n \frac{h_0^{(m)}(z_k, t, u)}{z_l^{j_l-1}} \sum_{\pi\in S_n} \mathrm{sgn}\,\pi \prod_{k=1}^n \{(1 + z_k)z_k\}^{\pi(k)-1}.$$

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Using the Vandermonde determinant $\sum_{\pi\in S_n}\operatorname{sgn}\pi\prod_{k=1}^n y_k^{\pi(k)-1}=\prod_{i<j}(y_j-y_i)$, we obtain this sum becomes

$$\operatorname{CT}_{\boldsymbol{z}}\prod_{k=1}^n h_0^{(m)}(z_k,t,u)\prod_{1\le i<j\le n}\{z_j(1+z_j)-z_i(1+z_i)\}\sum_{I=\{j_1,\dots,j_n\}}\alpha(I)\prod_{k=1}^n z_k^{-j_k+1}$$

$$=\operatorname{CT}_{\boldsymbol{z}}\prod_{k=1}^n h_0^{(m)}(z_k^{-1},t,u)\prod_{1\le i<j\le n}\{z_j^{-1}(1+z_j^{-1})-z_i^{-1}(1+z_i^{-1})\}\sum_{I=\{j_1,\dots,j_n\}}\alpha(I)\prod_{k=1}^n z_k^{j_k-1}.$$

This identity follows since the constant term is not changed by the transformation $z_l\to z_l^{-1}$. If we use the fact that $\operatorname{CT}_{\boldsymbol{z}}g(z_1,\cdots,z_n)=\frac{1}{n!}\operatorname{CT}_{\boldsymbol{z}}\sum_{\sigma\in S_n}g(z_{\sigma(1)},\cdots,z_{\sigma(n)})$ for any polynomial $g\in\mathbb{C}[z_1,\dots,z_n]$, then the above sum becomes

$$\frac{1}{n!}\operatorname{CT}_{\boldsymbol{z}}\prod_{k=1}^n h_0^{(m)}(z_k^{-1},t,u)\prod_{1\le i<j\le n}\{z_j^{-1}(1+z_j^{-1})-z_i^{-1}(1+z_i^{-1})\}$$

$$\times\sum_{\sigma\in S_n}\operatorname{sgn}\sigma\sum_{I=\{j_1,\dots,j_n\}}\alpha(I)\prod_{k=1}^n z_{\sigma(k)}^{j_k-1}.$$

Now use the minor summation formula (2.9) to obtain

$$\sum_{I=\{j_1,\dots,j_n\}\in\binom{[n+N]}{n}}(-1)^{s(\bar I,I)}\operatorname{Pf}\left(A_{\bar I}^{\bar I}\right)\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{k=1}^n z_{\sigma(k)}^{j_k-1}=G_A(z_1,\dots,z_n).\quad(8.21)$$

Substituting (8.21) into the above identity, we see that $D_{n,m}(A,t,u)$ is equal to

$$\frac{1}{n!}\operatorname{CT}_{\boldsymbol{z}}\prod_{k=1}^n h_0^{(m)}(z_k^{-1},t,u)\prod_{1\le i<j\le n}\{z_j^{-1}(1+z_j^{-1})-z_i^{-1}(1+z_i^{-1})\}G_A(z_1,\cdots,z_n)$$

which is equal to

$$\frac{1}{n!}\operatorname{CT}_{\boldsymbol{z}}\prod_{k=1}^n h_0^{(m)}(z_k^{-1},t,u)G_A(z_1,\cdots,z_n)\sum_{\pi\in S_n}\operatorname{sgn}(\pi)\prod_{k=1}^n\{z_k^{-1}(1+z_k^{-1})\}^{\pi(k)-1}$$

$$=\operatorname{CT}_{\boldsymbol{z}}\prod_{k=1}^n h_0^{(m)}(z_k^{-1},t,u)\prod_{k=1}^n\{z_k^{-1}(1+z_k^{-1})\}^{k-1}G_A(z_1,\cdots,z_n).$$

Now by changing all the index $k\to n+1-k$, we obtain

$$\operatorname{CT}_{\boldsymbol{z}}\prod_{k=1}^n h_0^{(m)}(t,z_k^{-1})\prod_{k=1}^n\{z_k^{-1}(1+z_k^{-1})\}^{n-k}F_A(z_1,\cdots,z_n)\prod_{1\le i<j\le n}(z_i-z_j)$$

$$=\operatorname{CT}_{\boldsymbol{z}}\prod_{k=1}^n h_0^{(m)}(t,z_k^{-1})\prod_{k=1}^n(1+z_k^{-1})^{n-k}\prod_{1\le i<j\le n}\left(1-\frac{z_j}{z_i}\right)F_A(z_1,\cdots,z_n).$$

This complete the proof. $\qquad\square$

9 Concluding remarks

First of all, we should note that the evaluations of the Pfaffians appearing in Section 7 are still open. Mills, Robbins and Rumsey had the simple forms of their conjectures in [26] when $m = 0$. In this paper we gave the Pfaffian forms and constant term expressions of their conjectures in Section 7 and Section 8. In particular the Pfaffians in Section 7 are approximately of size $(2n + m - 1)$. But it is also possible to make them into Pfaffians of size n or of size $n + 1$ using (2.12) or (2.14) (cf. [2, 18, 32]). Here we adopt the above Pfaffians since they do not depend on whether n is even or odd. To evaluate these Pfaffians or the constant terms, maybe one needs the other tools. In the forth coming paper [12] which will appear soon, we will study the other two conjectures by Mills, Robbins and Rumsey, (i.e. [26, Conjecture 4, Conjecture 6]).

References

[1] G.E. Andrews, "Pfaff's method (I): the Mills-Robbins-Rumsey determinant", *Discrete Math.* **193** (1998), 43–60.

[2] G.E. Andrews, "Plane partitions V: the TSSCPP conjecture", *J. Combin. Theory Ser. A* **66** (1994), 28–39.

[3] G.E. Andrews and W.H. Burge, "Determinant identities", *Pacific J. Math.* **158** (1993), 1–14.

[4] D.M. Bressound, *Proofs and Confirmations*, Cambridge U.P.

[5] W.F. Doran, "A connection between alternating sign matrices and totally symmetric self-complementary plane partitions", preprint.

[6] P. Di Francesco, "A refined Razumnov-Stroganov conjecture", `arXiv:cond-mat/0407477`.

[7] P. Di Francesco and P. Zinn-Justin, "Around the Razumov-Stroganov conjecture: proof of a multi-parameter sum rule", `arXiv:math-ph/0410061`, Electron. J. Combin. **12** (2005), R6.

[8] T. Eisenkölbl, "(-1)-enumeration of plane partitions with complementation symmetry" Adv. in Appl. Math. **30** (2003), 53–95.

[9] I. Gessel and G. Viennot, *Determinants, Paths, and Plane Partitions*, preprint (1989).

[10] T. Hashimoto, "A central element in the universal enveloping algebra of type D_n via minor summation formula of Pfaffians", `arXiv:math.RT/0602055`.

[11] M. Ishikawa, "Minor summation formula and a proof of Stanley's open problem", `arXiv:math.CO/0408204`.

[12] M. Ishikawa, "On refined enumerations of totally symmetric self-complementary plane partitions II", in preparation.

[13] M. Ishikawa, H. Tagawa, S. Okada and J. Zeng, "Generalizations of Cauchy's determinant and Schur's Pfaffian", `arXiv:math.CO/0411280`, to appear in *Adv. in Appl. Math.*

[14] M. Ishikawa, S. Okada and M. Wakayama, "Minor summation of Pfaffians and generalized Littlewood type formulas", *J. Alg.* **183** (1996), 193-216.

[15] M. Ishikawa and M. Wakayama, "Minor summation formula of Pfaffians", *Linear and Multilinear algebra* **39** (1995), 285–305.

[16] M. Ishikawa and M. Wakayama, "Applications of the minor summation formula III: Plücker relations, lattice pathes and Pfaffians", `arXiv:math.CO/0312358`, *J. Combin. Theory Ser. A* **113** (2006) 113-155.

[17] D.E. Knuth, "Overlapping Pfaffians", *Electron. J. Combin.* **3**, 151–163.

[18] C. Krattenthaler, "Determinant identities and a generalization of the number of totally symmetric self-complementary plane partitions", *Electron. J. Combin.* **4(1)** (1997), #R27.

[19] C. Krattenthaler, "Advanced determinant calculus", *Sem. Lothar. Combin.* **42** ("The Andrews Festschrift") (1999), Article B42q.

[20] C. Krattenthaler, a manuscript on the magog trapezoids (private communication).

[21] G. Kuperberg, "Another proof of the alternating-aign matrix conjecture", *Int. Math. Res. Not.* **3** (1996), 139–150. `arXiv:math.CO/9810091`.

[22] G. Kuperberg, "An exploration of the permanent-determinant method", *Electron. J. Combin.* **5** (1998), #R64, `arXiv:math.CO/9810091`.

[23] G. Kuperberg, "Symmetry classes of alternating-sign matrices under one roof", *Ann. of Math.* (2) **156** (2002), 835-866, `arXiv:math.CO/0008184`.

[24] I. G. Macdonald, *Symmetric Functions and Hall Polynomials (2nd ed.)*, Oxford Univ. Press, (1995).

[25] W.H. Mills, D.P. Robbins and H. Rumsey, "Alternating sign matrices and descending plane partitions", *J. Combin. Theory Ser. A* **34**, (1983), 340–359.

[26] W.H. Mills, D.P. Robbins and H. Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser. A* **42**, (1986), 277–292.

[27] S. Okada, "Enumeration of symmetry classes of alternating sign matrices and characters of classical groups", `arXiv:math.CO/0308234`, *J. Algebraic Combin.* **23** (2006), 43 – 69.

[28] D.P. Robbins, "Symmetry classes of alternating sign matrices", `arXiv:math.CO/0008045`.

[29] A.V. Razumov and Yu. G. Stroganov, "On refined enumerations of some symmetry classes of ASMs", `arXiv:math-ph/0312071`.

[30] R.P. Stanley, "Symmetries of plane partitions", *J. Combin. Theory Ser. A* **43**, (1986), 103–113.

[31] R.P. Stanley, *Enumerative combinatorics, Volume II*, Cambridge University Press, (1999).

[32] J.R. Stembridge, "Nonintersecting paths, Pfaffians, and plane partitions" *Adv. math.*, **83** (1990), 96–131.

[33] J.R. Stembridge, "Strange Enumerations of CSPP's and TSPP's", preprint.

[34] Yu.G . Stroganov, "A new way to deal with Izergin-Korepin determinant at root of unity" `arXiv:math-ph/0204042`.

[35] D. Zeilberger, "A constant term identity featuring the ubiquitous (and mysterious) Andrews-Mills-Robbins-Rumsey numbers", *J. Combin. Theory Ser. A* **66** (1994), 17–27.

[36] D. Zeilberger, "Proof of the refined alternating sign matrix conjecture", *New York J. Math.* **2** (1996), 59–68.
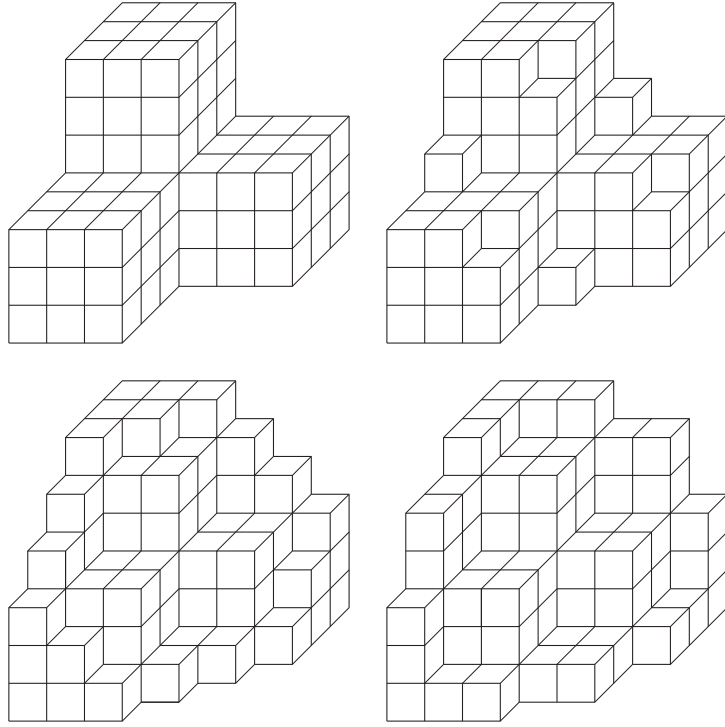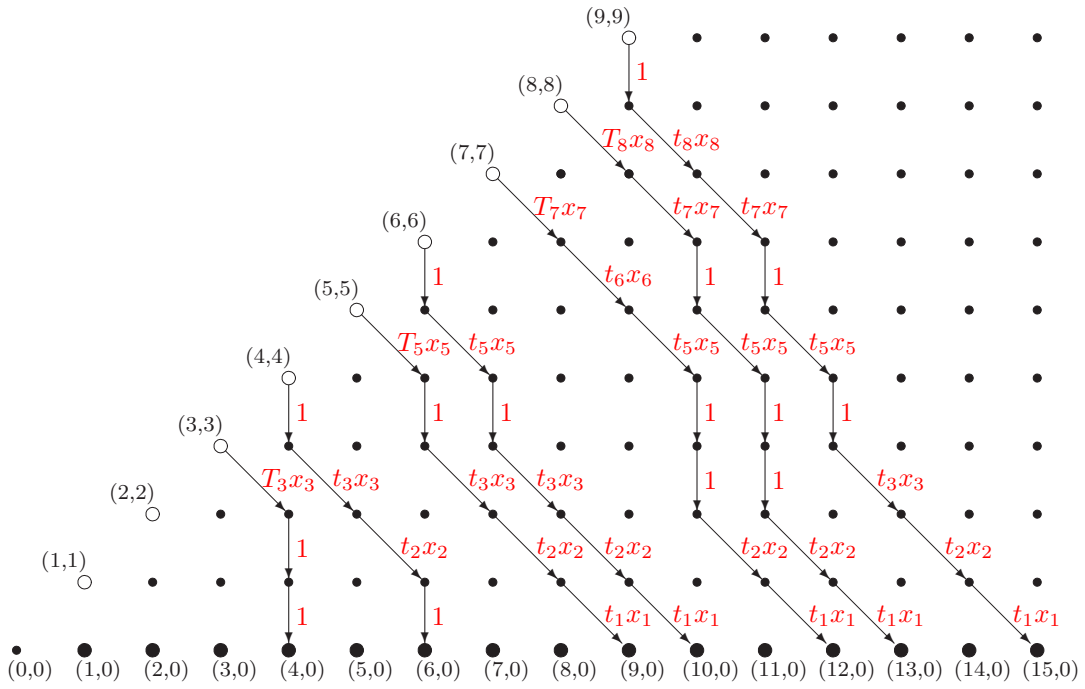
Figure 1: TSSCPP ($n = 1$, $m = 2$)

Figure 2: Lattice Paths ($n = 7$, $m = 3$, $\lambda' = (65^2 4^2 21)$, $T_i = \prod_{k=i}^{n+m} t_k$.)