A Proof of Stanley's Open Problem

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Abstract

In the open problem session of the FPSAC'03, R.P. Stanley gave an open problem about a certain sum of the Schur functions (See [19]). The purpose of this paper is to give a proof of this open problem. The proof consists of three steps. At the first step we express the sum by a Pfaffian as an application of our minor summation formula ([7]). In the second step we prove a Pfaffian analogue of Cauchy type identity which generalize [22]. Then we give a proof of Stanley's open problem in Section 4. We also present certain corollaries obtained from this identity involving the Big Schur functions and some polynomials arising from the Macdonald polynomials, which generalize Stanley's open problem.

Résumé

Dans la session de problèmes de SFCA'03, Stanley a posé un problème ouvert sur certaine somme de fonctions de Schur (voir [19]). Le but de cet article est de résoudre ce problème ouvert. La preuve consiste en trois étapes. Premièrement on exprime cette somme comme un Pfaffien en appliquant notre formule de sommation de mineurs [7]. Deuxièmement on démontre un analogue Pfaffien de l'identité de type Cauchy, qui généralise une identité de Sunquist [22]. Et puis on résoud le probème ouvert de Stanley dans la Section 4. On présente aussi quelques corollaires de cette identité impliquant les grandes fonctions de Schur et des polynômes apparaissant dans l'étude des polynômes de Macdonald, qui généralise le problème originel de Stanley.

1 Introduction

In the open problem session of the 15th Anniversary International Conference on Formal Power Series and Algebraic Combinatorics (Vadstena, Sweden, 25 June 2003), R.P. Stanley gave an open problem on a sum of Schur functions with a weight including four parameters, i.e. Theorem 1.1 (See [19]). The purpose of this paper is to give a proof of this open problem. In the process of our proof, we obtain a Pfaffian identity, i.e. Theorem 3.1, which generalize the Pfaffian identities in [22]. Note that certain determinant and Pfaffian identities of this type first appeared in [15], and applied to solve some alternating sign matrices enumerations under certain symmetries stated in [11]. Certain conjectures which extensively generalize the determinant and Pfaffian identities of this type were stated in [17], and a proof of the conjectured determinant and Pfaffian identities was given in [6]. Our proof proceeds by three steps. In the first step we utilize the minor summation formula ([7]) to express the sum of Schur functions into a Pfaffian. In the second step we express the Pfaffian by a determinant using a Cauchy type Pfaffian formula (also see [16], [17] and [6]), and try to simplify it as much as possible. In the final step we complete our proof using a key proposition, i.e. Proposition 4.1 (See [18] and [21]).

We follow the notation in [13] concerning the symmetric functions. In this paper we use a symmetric function f in n variables $(x_1, \dots, x_n)$, which is usually written as $f(x_1, \dots, x_n)$, and also a symmetric function f in countably many variables $x = (x_1, x_2, \dots)$, which is written as $f(x)$ (for detailed description of the ring of symmetric functions in countably many variables, see [13], I, sec.2). To simplify this notation we express the n-tuple $(x_1, \dots, x_n)$ by X_n, and sometimes simply write $f(X_n)$ for $f(x_1, \dots, x_n)$. When the number of variables is finite and there is no fear of confusion what this number is, we simply write X for X_n in abbreviation. Thus $f(x)$ is in countably many variables, but $f(X)$ is in finite variables and the number of variables is clear from the assumption.

Given a partition λ, define $\omega(\lambda)$ by

$$\omega(\lambda) = a^{\sum_{i\geq 1}\lceil \lambda_{2i-1}/2\rceil} b^{\sum_{i\geq 1}\lfloor \lambda_{2i-1}/2\rfloor} c^{\sum_{i\geq 1}\lceil \lambda_{2i}/2\rceil} d^{\sum_{i\geq 1}\lfloor \lambda_{2i}/2\rfloor},$$

where a, b, c and d are indeterminates, and $\lceil x\rceil$ (resp. $\lfloor x\rfloor$) stands for the smallest (resp. largest) integer greater (resp. less) than or equal to x for a given real number x. For example, if $\lambda = (5, 4, 4, 1)$ then $\omega(\lambda)$ is the product of the entries in the following diagram for λ.

$$\begin{array}{|c|c|c|c|c|}
\hline
a & b & a & b & a \\
\hline
c & d & c & d \\
\cline{1-4}
a & b & a & b \\
\cline{1-4}
c \\
\cline{1-1}
\end{array}$$

Let $s_\lambda(x)$ denote the Schur function corresponding to a partition λ. R. P. Stanley gave the following conjecture in the open problem session of FPSAC'03.

Theorem 1.1. Let

$$z = \sum_{\lambda} \omega(\lambda) s_\lambda.$$

Here the sum runs over all partitions λ. Then we have

$$\log z - \sum_{n\geq 1} \frac{1}{2n} a^n (b^n - c^n) p_{2n} - \sum_{n\geq 1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \dots]]. \tag{1.1}$$

Here $p_r = \sum_{i\geq 1} x_i^r$ denote the rth power sum symmetric function.

As direct consequence of this theorem, we obtain the following corollary. Let $S_\lambda(x;t) = \det\left(q_{\lambda_i - i + j}(x;t)\right)_{1\leq i,j\leq \ell(\lambda)}$ denote the big Schur function corresponding the partitions λ, where $q_r(x;t) = Q_{(r)}(x;t)$ denote the Hall-Littlewood functions (See [13], III, sec.2).

Corollary 1.2. Let

$$Z(x;t) = \sum_{\lambda} \omega(\lambda) S_\lambda(x;t),$$

Here the sum runs over all partitions λ. Then we have

$$\log Z(x;t) - \sum_{n\geq 1} \frac{1}{2n} a^n (b^n - c^n)(1-t^{2n}) p_{2n} - \sum_{n\geq 1} \frac{1}{4n} a^n b^n c^n d^n (1-t^{2n})^2 p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \dots]]. \tag{1.2}$$

This corollary is also generalized to the two parameter polynomials defined by I. G. Macdonald. Define

$$T_\lambda(x;q,t) = \det \left(Q_{(\lambda_i - i + j)}(x;q,t) \right)_{1\leq i,j\leq \ell(\lambda)}$$

where $Q_\lambda(x;q,t)$ stands for the Macdonald polynomial corresponding to the partition λ, and $Q_{(r)}(x;q,t)$ is the one corresponding to the one row partition (r) (See [13], IV, sec.4). Then we obtain the following corollary:

Corollary 1.3. Let

$$Z(x;q,t) = \sum_\lambda \omega(\lambda) T_\lambda(x;q,t),$$

Here the sum runs over all partitions λ. Then we have

$$\log Z(x;q,t) - \sum_{n\geq 1} \frac{1}{2n} a^n (b^n - c^n) \frac{1-t^{2n}}{1-q^{2n}} p_{2n} - \sum_{n\geq 1} \frac{1}{4n} a^n b^n c^n d^n \frac{(1-t^{2n})^2}{(1-q^{2n})^2} p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \dots]]. \tag{1.3}$$

In the rest of this section we briefly recall the definition of Pfaffians. For the detailed explanation of Pfaffians, the reader can consult [9] and [20]. Let n be a non-negative integer and assume we are given a $2n$ by $2n$ skew-symmetric matrix $A = (a_{ij})_{1\leq i,j\leq 2n}$, (i.e. $a_{ji} = -a_{ij}$), whose entries a_{ij} are in a commutative ring. The *Pfaffian* of A is, by definition,

$$\operatorname{Pf}(A) = \sum \epsilon(\sigma_1, \sigma_2, \dots, \sigma_{2n-1}, \sigma_{2n})\, a_{\sigma_1\sigma_2} \dots a_{\sigma_{2n-1}\sigma_{2n}}.$$

where the summation is over all partitions $\{\{\sigma_1, \sigma_2\}_<, \dots, \{\sigma_{2n-1}, \sigma_{2n}\}_<\}$ of $[2n]$ into 2-elements blocks, and where $\epsilon(\sigma_1, \sigma_2, \dots, \sigma_{2n-1}, \sigma_{2n})$ denotes the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & 2n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{2n} \end{pmatrix}.$$

2 Pfaffian Expressions

Given a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ satisfying $\ell(\lambda) \leq m$, we associate a decreasing sequence $\lambda + \delta_m$ which is usually denoted by $l = (l_1, \dots, l_m)$, where $\delta_m = (m-1, m-2, \dots, 0)$. The key observation to prove Theorem 1.1 is the following theorem, which shows that the weight $\omega(\lambda)$ can be expressed by a Pfaffian:

Theorem 2.1. Let n be a non-negative integer. Let $\lambda = (\lambda_1, \dots, \lambda_{2n})$ be a partition such that $\ell(\lambda) \leq 2n$, and put $l = (l_1, \dots, l_{2n}) = \lambda + \delta_{2n}$. Define a $2n$ by $2n$ skew-symmetric matrix $A = (\alpha_{ij})_{1\leq i,j\leq 2n}$ by

$$\alpha_{ij} = a^{\lceil (l_i - 1)/2 \rceil} b^{\lfloor (l_i - 1)/2 \rfloor} c^{\lceil l_j/2 \rceil} d^{\lfloor l_j/2 \rfloor}$$

for $i < j$, and as $\alpha_{ji} = -\alpha_{ij}$ holds for any $i, j \geq 0$. Then we have

$$\operatorname{Pf}\left[A\right]_{1\leq i,j\leq 2n} = (abcd)^{\binom{n}{2}} \omega(\lambda).$$

The essential idea to prove Theorem 2.1 is the following lemma which already appeared as Lemma 7 in Section 4 of [7].

Lemma 2.2. Let x_i and y_j be indeterminates, and let n is a non-negative integer. Then

$$\mathrm{Pf}[x_i y_j]_{1\le i<j\le 2n} = \prod_{i=1}^{n} x_{2i-1} \prod_{i=1}^{n} y_{2i}. \ \square \tag{2.1}$$

Theorem 2.1 shows that the weight $\omega(\lambda)$ can be expressed by the Pfaffians of submatrices of a certain matrix, and the row/column indices of the submatrices are determined by the partition λ. This shows that the weighted sum of the Schur functions is a sum of minors multiplied by the "sub-Pfaffians". Thus we need a minor summation formula from [7].

Let m, n and r be integers such that $r \le m, n$ and let T be an m by n matrix. For any index sets $I = \{i_1, \dots, i_r\}_< \subseteq [m]$ and $J = \{j_1, \dots, j_r\}_< \subseteq [n]$, let $\Delta^I_J(A)$ denote the sub-matrix obtained by selecting the rows indexed by I and the columns indexed by J. If $r = m$ and $I = [m]$, we simply write $\Delta_J(A)$ for $\Delta^{[m]}_J(A)$. Similarly, if $r = n$ and $J = [n]$, we write $\Delta^I(A)$ for $\Delta^I_{[n]}(A)$. For any finite set S and a non-negative integer r, let $\binom{S}{r}$ denote the set of all r-element subsets of S. We cite a theorem from [7] which we call a minor summation formula:

Theorem 2.3. Let n and N be non-negative integers such that $2n \le N$. Let $T = (t_{ij})_{1\le i\le 2n, 1\le j\le N}$ be a $2n$ by N rectangular matrix, and let $A = (a_{ij})_{1\le i,j\le N}$ be a skew-symmetric matrix of size N. Then

$$\sum_{I\in\binom{[N]}{2n}} \mathrm{Pf}\left(\Delta^I_I(A)\right) \det\left(\Delta_I(T)\right) = \mathrm{Pf}\left(T A\, {}^t T\right).$$

If we put $Q = (Q_{ij})_{1\le i,j\le 2n} = T A\, {}^t T$, then its entries are given by

$$Q_{ij} = \sum_{1\le k<l\le N} a_{kl} \det\left(\Delta^{ij}_{kl}(T)\right), \qquad (1 \le i, j \le 2n).$$

Here we write $\Delta^{ij}_{kl}(T)$ for $\Delta^{\{ij\}}_{\{kl\}}(T) = \begin{vmatrix} t_{ik} & t_{il} \\ t_{jk} & t_{jl} \end{vmatrix}$. $\square$

First we restrict our attention to the finite variables case. As an application of the minor summation formula, i.e. Theorem 2.3, we can express the sum with a Pfaffian.

Theorem 2.4. Let n be a positive integer and let $\omega(\lambda)$ be as defined in Section 1. Let

$$z_n = z_n(X_{2n}) = \sum_{\ell(\lambda)\le 2n} \omega(\lambda) s_\lambda(X_{2n}) = \sum_{\ell(\lambda)\le 2n} \omega(\lambda) s_\lambda(x_1, \dots, x_{2n}) \tag{2.2}$$

be the sum restricted to $2n$ variables. Then we have

$$z_n(X_{2n}) = \frac{(abcd)^{-\binom{n}{2}}}{\prod_{1\le i<j\le 2n}(x_i - x_j)} \mathrm{Pf}\left(p_{ij}\right)_{1\le i<j\le 2n}, \tag{2.3}$$

where p_{ij} is defined by

$$p_{ij} = \frac{\begin{vmatrix} x_i + a x_i^2 & 1 - a(b+c)x_i - abc x_i^3 \\ x_j + a x_j^2 & 1 - a(b+c)x_j - abc x_j^3 \end{vmatrix}}{(1 - abx_i^2)(1 - abx_j^2)(1 - abcd x_i^2 x_j^2)}. \tag{2.4}$$

Proof. By Theorem 2.3 it is enough to compute

$$\beta_{ij} = \sum_{k \geq l \geq 0} a^{\lceil (k-1)/2 \rceil} b^{\lfloor (k-1)/2 \rfloor} c^{\lceil l/2 \rceil} d^{\lfloor l/2 \rfloor} \begin{vmatrix} x_i^k & x_i^l \\ x_j^k & x_j^l \end{vmatrix}.$$

Let $f_{kl}^{ij} = a^{\lceil (k-1)/2 \rceil} b^{\lfloor (k-1)/2 \rfloor} c^{\lceil l/2 \rceil} d^{\lfloor l/2 \rfloor} \begin{vmatrix} x_i^k & x_i^l \\ x_j^k & x_j^l \end{vmatrix}$, then, this sum can be divided into four cases, i.e.

$$\beta_{ij} = \sum_{\substack{k=2r+1,\, l=2s \\ r \geq s \geq 0}} f_{kl}^{ij} + \sum_{\substack{k=2r,\, l=2s \\ r \geq s \geq 0}} f_{kl}^{ij} + \sum_{\substack{k=2r+1,\, l=2s+1 \\ r \geq s \geq 0}} f_{kl}^{ij} + \sum_{\substack{k=2r+2,\, l=2s+1 \\ r \geq s \geq 0}} f_{kl}^{ij}.$$

We compute each case:

(i) If $k = 2r + 1$ and $l = 2s$ for $r \geq s \geq 0$, then

$$\begin{aligned} \sum_{\substack{k=2r+1,\, l=2s \\ r \geq s \geq 0}} f_{kl}^{ij} &= \sum_{r \geq s \geq 0} a^r b^r c^s d^s \begin{vmatrix} x_i^{2r+1} & x_i^{2s} \\ x_j^{2r+1} & x_j^{2s} \end{vmatrix} \\ &= \sum_{r \geq s \geq 0} c^s d^s \begin{vmatrix} \frac{a^s b^s x_i^{2s+1}}{1-abx_i^2} & x_i^{2s} \\ \frac{a^s b^s x_j^{2s+1}}{1-abx_j^2} & x_j^{2s} \end{vmatrix} \\ &= \frac{(x_i - x_j)(1 + abx_i x_j)}{(1 - abx_i^2)(1 - abx_j^2)(1 - abcdx_i^2 x_j^2)}. \end{aligned}$$

In the same way we obtain the followings by straight forward computations.

(ii) If $k = 2r$ and $l = 2s$ for $r \geq s \geq 0$, then

$$\sum_{\substack{k=2r,\, l=2s \\ r \geq s \geq 0}} f_{kl}^{ij} = \frac{a(x_i^2 - x_j^2)}{(1 - abx_i^2)(1 - abx_j^2)(1 - abcdx_i^2 x_j^2)}.$$

(iii) If $k = 2r + 1$ and $l = 2s + 1$ for $r \geq s \geq 0$, then

$$\sum_{\substack{k=2r+1,\, l=2s+1 \\ r \geq s \geq 0}} f_{kl}^{ij} = \frac{abcx_i x_j (x_i^2 - x_j^2)}{(1 - abx_i^2)(1 - abx_j^2)(1 - abcdx_i^2 x_j^2)}.$$

(iv) If $k = 2r + 2$ and $l = 2s + 1$ for $r \geq s \geq 0$, then

$$\sum_{\substack{k=2r+2,\, l=2s+1 \\ r \geq s \geq 0}} f_{kl}^{ij} = \frac{acx_i x_j (x_i - x_j)(1 + abx_i x_j)}{(1 - abx_i^2)(1 - abx_j^2)(1 - abcdx_i^2 x_j^2)}.$$

Summing up these four identities, we obtain

$$\beta_{ij} = \frac{(x_i - x_j)\{1 + abx_i x_j + a(x_i + x_j) + abcx_i x_j (x_i + x_j) + acx_i x_j (1 + abx_i x_j)\}}{(1 - abx_i^2)(1 - abx_j^2)(1 - abcdx_i^2 x_j^2)}.$$

It is easy to see the numerator is written by the determinant, and this completes the proof. $\square$

3 Cauchy Type Pfaffians

The aim of this section is to prove (3.4). In the next section we will use this identity to prove Stanley's open problem. First we prove a fundamental Pfaffian identity, i.e. Theorem 3.1, and deduce all the identities in this section to this theorem. An intensive generalization was conjectured in [17] and proved in [6]. There is a certain Pfaffian-Hafnian analogue of Borchardt's identity in [5].

First we fix our notation. Let n be an non-negative integer. Let $X = (x_1, \dots, x_{2n})$, $Y = (y_1, \dots, y_{2n})$, $A = (a_1, \dots, a_{2n})$ and $B = (b_1, \dots, b_{2n})$ be $2n$-tuples of variables. Set $V^n_{ij}(X, Y; A, B)$ to be

$$\begin{cases} a_i x_i^{n-j} y_i^{j-1} & \text{if } 1 \le j \le n, \\ b_i x_i^{2n-j} y_i^{j-n-1} & \text{if } n+1 \le j \le 2n, \end{cases}$$

for $1 \le i \le 2n$, and define $V^n(X, Y; A, B)$ by

$$V^n(X, Y; A, B) = \det \left(V^n_{ij}(X, Y; A, B) \right)_{1 \le i,j \le 2n}.$$

For example, if $n = 1$, then we have $V^1(X, Y; A, B) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, and if $n = 2$, then $V^2(X, Y; A, B)$ looks as follows:

$$V^2(X, Y; A, B) = \begin{vmatrix} a_1x_1 & a_1y_1 & b_1x_1 & b_1y_1 \\ a_2x_2 & a_2y_2 & b_2x_2 & b_2y_2 \\ a_3x_3 & a_3y_3 & b_3x_3 & b_3y_3 \\ a_4x_4 & a_4y_4 & b_4x_4 & b_4y_4 \end{vmatrix}.$$

The main result of this section is the following theorem.

Theorem 3.1. Let n be a positive integer. Let $X = (x_1, \dots, x_{2n})$, $Y = (y_1, \dots, y_{2n})$, $A = (a_1, \dots, a_{2n})$, $B = (b_1, \dots, b_{2n})$, $C = (c_1, \dots, c_{2n})$ and $D = (d_1, \dots, d_{2n})$ be $2n$-tuples of variables. Then

$$\mathrm{Pf} \left[\frac{\begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} \cdot \begin{vmatrix} c_i & d_i \\ c_j & d_j \end{vmatrix}}{\begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix}} \right]_{1 \le i < j \le 2n} = \frac{V^n(X, Y; A, B) V^n(X, Y; C, D)}{\prod_{1 \le i < j \le 2n} \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix}}. \tag{3.1}$$

The following proposition is obtained easily by elementary transformations of the matrices and we omit the proof.

Proposition 3.2. Let n be a positive integer. Let $X = (x_1, \dots, x_{2n})$ be $2n$-tuples of variables and let t be an indeterminate. Then

$$V^n(X, \mathbf{1} + tX^2; X, \mathbf{1}) = (-1)^{\binom{n}{2}} t^{\binom{n}{2}} \prod_{1 \le i < j \le 2n} (x_i - x_j), \tag{3.2}$$

where $\mathbf{1}$ denotes the $2n$-tuple $(1, \dots, 1)$, and $\mathbf{1} + tX^2$ denotes the $2n$-tuple $(1 + tx_1^2, \dots, 1 + tx_{2n}^2)$.

Let t be an arbitrary indeterminate. If we set $y_i = 1 + tx_i^2$ in (3.1), then

$$\begin{vmatrix} x_i & 1 + tx_i^2 \\ x_j & 1 + tx_j^2 \end{vmatrix} = (x_i - x_j)(1 - tx_ix_j)$$

and (3.2) immediately implies the following corollary.

Corollary 3.3. Let n be a non-negative integer. Let $X = (x_1, \dots, x_{2n})$, $A = (a_1, \dots, a_{2n})$, $B = (b_1, \dots, b_{2n})$, $C = (c_1, \dots, c_{2n})$ and $D = (d_1, \dots, d_{2n})$ be $2n$-tuples of variables. Then

$$\operatorname{Pf}\left[\frac{(a_ib_j - a_jb_i)(c_id_j - c_jd_i)}{(x_i - x_j)(1 - tx_ix_j)}\right]_{1 \le i < j \le 2n} = \frac{V^n(X, \mathbf{1} + tX^2; A, B)V^n(X, \mathbf{1} + tX^2; C, D)}{\prod_{1 \le i < j \le 2n}(x_i - x_j)(1 - tx_ix_j)}. \tag{3.3}$$

In particular, we have

$$\operatorname{Pf}\left[\frac{a_ib_j - a_jb_i}{1 - tx_ix_j}\right]_{1 \le i < j \le 2n} = (-1)^{\binom{n}{2}} t^{\binom{n}{2}} \frac{V^n(X, \mathbf{1} + tX^2; A, B)}{\prod_{1 \le i < j \le 2n}(1 - tx_ix_j)}. \square \tag{3.4}$$

Now we give a sketch of a proof of Theorem 3.1. Let n and r be integers such that $2n \ge r \ge 0$. Let $X = (x_1, \dots, x_{2n})$ be a $2n$-tuple of variables and let $1 \le k_1 < \dots < k_r \le 2n$ be a sequence of integers. Let $X^{(k_1,\dots,k_r)}$ denote the $(2n - r)$-tuple of variables obtained by removing the variables $x_{k_1}, \dots, x_{k_r}$ from X_{2n}. The key to prove Theorem 3.1 is the following lemma:

Lemma 3.4. Let n be a positive integer. Let $X = (x_1, \dots, x_{2n})$, $A = (a_1, \dots, a_{2n})$ and $C = (c_1, \dots, c_{2n})$ be $2n$-tuples of variables. Then the following identity holds.

$$\sum_{k=1}^{2n-1} \frac{\prod_{\substack{i=1\\ i \ne k}}^{2n-1}(x_k - x_i)}{x_k - x_{2n}}(a_k - a_{2n})(c_k - c_{2n})$$
$$\times V^{n-1}(X^{(k,2n)}, \mathbf{1}^{(k,2n)}; A^{(k,2n)}, \mathbf{1}^{(k,2n)})V^{n-1}(X^{(k,2n)}, \mathbf{1}^{(k,2n)}; C^{(k,2n)}, \mathbf{1}^{(k,2n)})$$
$$= \frac{V^n(X, \mathbf{1}; A, \mathbf{1})V^n(X, \mathbf{1}; C, \mathbf{1})}{\prod_{i=1}^{2n-1}(x_i - x_{2n})}.$$

Here $\mathbf{1}$ denotes the $2n$-tuples $(1, \dots, 1)$.

This lemma and the expansion of the Pfaffians along the last row/column implies Theorem 3.1 by a direct computation.

4 A Proof of Stanley's Open Problem

The key idea of our proof is the following proposition, which the reader can find in [18], Exercise 7.7, or [21], Section 3.

Proposition 4.1. Let $f(x_1, x_2, \dots)$ be a symmetric function with infinite variables. Then $f \in \mathbb{Q}[p_\lambda :$ all parts $\lambda_i > 0$ are odd$]$ if and only if

$$f(t, -t, x_1, x_2, \dots) = f(x_1, x_2, \dots). \square$$

Our strategy is simple. If we set $v_n(X_{2n})$ to be

$$\log z_n(X_{2n}) - \sum_{k \ge 1} \frac{1}{2k} a^k (b^k - c^k) p_{2k}(X_{2n}) - \sum_{k \ge 1} \frac{1}{4k} a^k b^k c^k d^k p_{2k}(X_{2n})^2 \tag{4.1}$$

then we claim it satisfies

$$v_{n+1}(t, -t, X_{2n}) = v_n(X_{2n}). \tag{4.2}$$

This will eventually prove Theorem 1.1. As an immediate consequence of (2.3), (2.4) and (3.4), we obtain the following theorem:

Theorem 4.2. Let $X=(x_1,\dots,x_{2n})$ be a $2n$-tuple of variables. Then

$$z_n(X_{2n})=(-1)^{\binom{n}{2}}\frac{V^n(X^2,\mathbf{1}+abcdX^4;X+aX^2,\mathbf{1}-a(b+c)X^2-abcX^3)}{\prod_{i=1}^{2n}(1-abx_i^2)\prod_{1\le i<j\le 2n}(x_i-x_j)(1-abcdx_i^2x_j^2)}, \tag{4.3}$$

where $X^2=(x_1^2,\dots,x_{2n}^2)$, $\mathbf{1}+abcdX^4=(1+abcdx_1^4,\dots,1+abcdx_{2n}^4)$, $X+aX^2=(x_1+ax_1^2,\dots,x_{2n}+ax_{2n}^2)$ and $\mathbf{1}-a(b+c)X^2-abcX^3=(1-a(b+c)x_1^2-abcx_1^3,\dots,1-a(b+c)x_{2n}^2-abcx_{2n}^3)$. □

The (4.3) is key expression to prove that $v_n(X_{2n})$ satisfies (4.2). Once one knows (4.3), then it is straightforward computation to prove Stanley's open problem. The following proposition is the first step.

Proposition 4.3. Let $X=(x_1,\dots,x_{2n})$ be a $2n$-tuple of variables. Put

$$f_n(X_{2n})=V^n(X^2,\mathbf{1}+abcdX^4;X+aX^2,\mathbf{1}-a(b+c)X^2-abcX^3).$$

Then $f_n(X_{2n})$ satisfies

$$\begin{aligned}&f_{n+1}(t,-t,X_{2n})\\&=(-1)^n 2t(1-abt^2)(1-act^2)\prod_{i=1}^{2n}(t^2-x_i^2)\prod_{i=1}^{2n}(1-abcdt^2x_i^2)\cdot f_n(X_{2n}).\end{aligned} \tag{4.4}$$

From Theorem 4.2 and Proposition 4.3 we obtain the following proposition.

Proposition 4.4. Let $X=(x_1,\dots,x_{2n})$ be a $2n$-tuple of variables. Then

$$z_{n+1}(t,-t,X_{2n})=\frac{1-act^2}{(1-abt^2)(1-abcdt^4)\prod_{i=1}^{2n}(1-abcdt^2x_i^2)}z_n(X_{2n}). \tag{4.5}$$

Now the proof of Theorem 1.1 is straightforward computation. We omit the details.

5 Open Problems

The author tried to find an analogous formula when the sum runs over all distinct partitions by computer experiments using Stembridge's SF package (cf. [2] and [3]). But the author could not find any conceivable formula when the sum runs over all distinct partitions.

He also checked Hall-Littlewood functions case, and could not find in the general case, but found some nice formulas if we substitute -1 into t. These are byproducts found by our computer experiments.

Conjecture 5.1. Let

$$w(x;t)=\sum_{\lambda}\omega(\lambda)P_\lambda(x;t),$$

where $P_\lambda(x;t)$ denote the Hall-Littlewood function corresponding to the partition λ, and the sum runs over all partitions λ. Then

$$\log w(x;-1)+\sum_{n\ge1\text{ odd}}\frac{1}{2n}a^nc^np_{2n}+\sum_{n\ge2\text{ even}}\frac{1}{2n}a^{\frac n2}c^{\frac n2}(a^{\frac n2}c^{\frac n2}-2b^{\frac n2}d^{\frac n2})p_{2n}\in\mathbb{Q}[[p_1,p_3,p_5,\dots]].$$

would hold. □

We might replace the Hall-Littlewood functions $P_\lambda(x;t)$ by the Macdonald polynomials $P_\lambda(x;q,t)$ in this conjecture. Let $P_\lambda(x;q,t)$ denote the Macdonald polynomial corresponding to the partition λ (See [13], IV, sec.4).

Conjecture 5.2. Let

$$w(x;q,t)=\sum_{\lambda}\omega(\lambda)P_\lambda(x;q,t).$$

Here the sum runs over all partitions λ. Then

$$\log w(x;q,-1)+\sum_{n\geq 1\text{ odd}}\frac{1}{2n}a^nc^np_{2n}+\sum_{n\geq 2\text{ even}}\frac{1}{2n}a^{\frac{n}{2}}c^{\frac{n}{2}}(a^{\frac{n}{2}}c^{\frac{n}{2}}-2b^{\frac{n}{2}}d^{\frac{n}{2}})p_{2n}$$
$$\in\mathbb{Q}[[p_1,p_3,p_5,\dots]]$$

would hold. □

6 Four-Parameter Partition Identities

In [2] C.E. Boulet gave a bijective proof of the following partition identities, i.e. Theorem 6.1 and Theorem 6.7. The aim of this section is to give another proof of these identities. To make our arguments easier, we first consider the strict partitions case.

Theorem 6.1. (Boulet)

$$\sum_{\mu\text{ strict partitions}}\omega(\mu)=\prod_{j=1}^{\infty}\frac{(1+a^jb^{j-1}c^{j-1}d^{j-1})(1+a^jb^jc^jd^{j-1})}{1-a^jb^jc^{j-1}d^{j-1}}.\qquad(6.1)$$

Here the sum runs over all strict partitions μ.

To prove this theorem, we need the following lemma, which can be derived from Lemma 2.2 by exactly the same method as we proved Lemma 2.1. Note that any strict partition μ can be written as $\mu_1>\cdots>\mu_{2n}\geq 0$ for a uniquely determined integer n. Let $\ell(\mu)$ denote the length of the strict partition μ, which is the number of nonzero parts of μ. For example, the length of $\mu=(10,8,7,5,3)$ is five.

Lemma 6.2. Let n be a nonnegative integer. Let $\mu=(\mu_1,\dots,\mu_{2n})$ be a strict partition such that $\mu_1>\cdots>\mu_{2n}\geq 0$. Define a skew-symmetric matrix $A=(\alpha_{ij})_{1\leq i,j\leq 2n}$ by

$$\alpha_{ij}=\begin{cases}a^{\lceil\mu_i/2\rceil}b^{\lfloor\mu_i/2\rfloor}c^{\lceil\mu_j/2\rceil}d^{\lfloor\mu_j/2\rfloor}z, & \text{if }\mu_j=0,\\ a^{\lceil\mu_i/2\rceil}b^{\lfloor\mu_i/2\rfloor}c^{\lceil\mu_j/2\rceil}d^{\lfloor\mu_j/2\rfloor}z^2, & \text{if }\mu_j>0,\end{cases}$$

for $i<j$, and as $\alpha_{ji}=-\alpha_{ij}$ holds for any $i,j\geq 0$. Then we have

$$\operatorname{Pf}[A]=\omega(\mu)z^{\ell(\mu)}.\ \square$$

Let J_n denote the square matrix of size n whose (i,j)-entry is $\delta_{i,n+1-j}$. We simply write J for J_n when there is no fear of confusion on the size n. The following lemma can be obtained from Theorem 3 of Section 3 in [7]. (cf. Theorem of Section 4 in [23]). The prototype of this type identity first appeared in [14].

Lemma 6.3. Let n be a positive integer. Let $A = (a_{ij})_{1\le i,j\le n}$ and $B = (b_{ij})_{1\le i,j\le n}$ be skew symmetric matrices of size n. Then

$$\sum_{t=0}^{r} z^r \sum_{I\in\binom{[n]}{2t}} \gamma^{|I|} \operatorname{Pf}\left(\Delta_I^I(A)\right) \operatorname{Pf}\left(\Delta_I^I(B)\right) = \operatorname{Pf}\begin{bmatrix} J\,{}^tAJ & J \\ -J & C \end{bmatrix}, \tag{6.2}$$

where $|I| = \sum_{i\in I} i$ and $C = (C_{ij})_{1\le i,j}$ is given by

$$C_{ij} = \gamma^{i+j} b_{ij} z.$$

Theorem 6.4. Let n be a positive integer. Then

$$\sum_{\substack{\mu \text{ strict partitions} \\ \mu_1 \le n}} \omega(\mu) z^{\ell(\mu)} = \operatorname{Pf}\begin{bmatrix} S & J_{n+1} \\ -J_{n+1} & B \end{bmatrix}, \tag{6.3}$$

where $S = (1)_{0\le i<j\le n}$ and $B = (\beta_{ij})_{0\le i<j\le n}$ with

$$\beta_{ij} = \begin{cases} a^{\lceil j/2\rceil} b^{\lfloor j/2\rfloor} c^{\lceil i/2\rceil} d^{\lfloor i/2\rfloor} z & \text{if } 0 = i < j \le n, \\ a^{\lceil j/2\rceil} b^{\lfloor j/2\rfloor} c^{\lceil i/2\rceil} d^{\lfloor i/2\rfloor} z^2 & \text{if } 0 < i < j \le n. \end{cases}$$

For example, if $n = 3$, then the Pfaffian in the right-hand side of (6.3) is

$$\operatorname{Pf}\left[\begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & az & abz & a^2bz \\ 0 & 0 & -1 & 0 & -az & 0 & abcz^2 & a^2bcz^2 \\ 0 & -1 & 0 & 0 & -abz & -abcz^2 & 0 & a^2bcdz^2 \\ -1 & 0 & 0 & 0 & -a^2bz & -a^2bcz^2 & -a^2bcdz^2 & 0 \end{array}\right],$$

and this is equal to $1 + a(1 + b + ab)z + abc(1 + a + ad)z^2 + a^3bcdz^3$. Meanwhile, the only strict partition such that $\ell(\mu) = 0$ is $\emptyset$, the strict partitions μ such that $\ell(\mu) = 1$ and $\mu_1 \le 3$ are the following three:

$$\begin{array}{|c|}\hline a \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline a & b \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline a & b & a \\ \hline\end{array}$$

the strict partitions μ such that $\ell(\mu) = 2$ and $\mu_1 \le 3$ are the following three:

$$\begin{array}{|c|c|}\hline a & b \\ \hline \multicolumn{1}{c|}{} & c \\ \cline{2-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline a & b & a \\ \hline \multicolumn{1}{c|}{} & c & \multicolumn{1}{|c}{} \\ \cline{2-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline a & b & a \\ \hline \multicolumn{1}{c|}{} & c & d \\ \cline{2-3}\end{array}$$

and the strict partition μ such that $\ell(\mu) = 4$ and $\mu_1 \le 3$ is the following one:

$$\begin{array}{|c|c|c|}\hline a & b & a \\ \hline \multicolumn{1}{c|}{} & c & d \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & a \\ \cline{3-3}\end{array}$$

The sum of the weights of these strict partitions correspond to the above Pfaffian.

Let $\psi_n = \psi_n(a, b, c, d; z) = \operatorname{Pf}\begin{bmatrix} S & J_{n+1} \\ -J_{n+1} & B \end{bmatrix}$ denote the right-hand side of (6.3) for a nonnegative integer n. For example, we have $\psi_0 = 1$, $\psi_1 = 1+az$, $\psi_2 = 1+a(1+b)z+abcz^2$ and $\psi_3 = 1+a(1+b+ab)z+abc(1+a+ad)z^2+a^3bcdz^3$. By elementary transformations and expansions along rows/columns of Pfaffians, we obtain the following recursion formula.

Proposition 6.5. Let $\psi_n = \psi_n(a, b, c, d; z)$ be as above. Then we have

$$\psi_{2n} = (1 + b)\psi_{2n-1} + (a^n b^n c^n d^{n-1} z^2 - b)\psi_{2n-2},$$
$$\psi_{2n+1} = (1 + a)\psi_{2n} + (a^{n+1} b^n c^n d^n z^2 - b)\psi_{2n-1},$$

for any positive integer j.

From this recurrence relation we immediately obtain the following corollary.

Corollary 6.6. Set $q = abcd$, $x_n = \psi_{2n}$ and $y_n = \psi_{2n+1}$ then

$$x_{n+1} = \left\{1 + ab + a(1 + bc)z^2 q^n\right\} x_n - ab(1 - z^2 q^n)(1 - acz^2 q^{n-1})x_{n-1},$$
$$y_{n+1} = \left\{1 + ab + abc(1 + ad)z^2 q^n\right\} y_n - ab(1 - z^2 q^n)(1 - acz^2 q^n)y_{n-1},$$

where $x_0 = 1$, $y_0 = 1 + az$, $x_1 = 1 + a(1 + b)z + abcz^2$ and

$$y_1 = 1 + a(1 + b + ab)z + abc(1 + a + ad)z^2 + a^3 bcdz^3.$$

There is no enough space here to describe the details of the results, but, when $z = 1$, we can identify them with the three-term relation of the Al-Salam Chihara polynomials and the solution is expressed by an appropriate basic hypergeometric series, i.e.

$$x_n(a, b, c, d; 1) = (-a; q)_n\, {}_2\phi_1\left(\begin{matrix} q^{-n}, -c \\ -a^{-1}q^{-n+1} \end{matrix} \,\middle|\, q; -bq\right),$$
$$y_n(a, b, c, d; 1) = (1 + a)(-abc; q)_n\, {}_2\phi_1\left(\begin{matrix} q^{-n}, -acd \\ -(abc)^{-1}q^{-n+1} \end{matrix} \,\middle|\, q; -c^{-1}q\right).$$

This method works to prove Theorem 6.1.

Finally let us mention that a similar argument also works to prove the following ordinary partition identity.

Theorem 6.7. (Boulet)

$$\sum_{\lambda \text{ partitions}} \omega(\lambda) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^{j-1} c^{j-1} d^{j-1})(1 + a^j b^j c^j d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^j c^{j-1} d^{j-1})(1 - a^j b^{j-1} c^j d^{j-1})}. \tag{6.4}$$

Here the sum runs over all partitions λ.

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