Selberg integrals and Catalan-Pfaffian Hankel determinants

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De Bruijn's formula

Let n be a positive integer, and let $\phi_i(x)$ and $\psi_i(x)$ be functions on $[0,a]$ for $1 \le i \le 2n$. Then

$$\int \cdots \int_{0\le x_1<\cdots<x_n\le a} \det\left(\phi_i(x_j)|\psi_i(x_j)\right)\, d_q\mu(x_1)\dots d_q\mu(x_n) = \operatorname{Pf}\left(Q_{i,j}\right)_{1\le i,j\le 2n},$$

where

$$Q_{i,j} = \int_0^a \left\{\phi_i(x)\psi_j(x) - \phi_j(x)\psi_i(x)\right\}\, d_q\mu(x)$$

and $(\phi_i(x_j)|\psi_i(x_j))$ denotes the $2n\times 2n$ matrix whose ith row is

$$(\phi_i(x_1),\psi_i(x_1),\dots,\phi_i(x_n),\psi_i(x_n))$$

for $1\le i\le 2n$.

Notation

Throughout this paper we use the standard notation for q-series

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \qquad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

for any integer n. Usually $(a;q)_n$ is called the q-shifted factorial, and we frequently use the compact notation:

$$(a_1,a_2,\dots,a_r;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n.$$

The ${}_r\phi_s$ basic hypergeometric series is defined by

$${}_r\phi_s\left(\begin{matrix}a_1,a_2,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,z\right) = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_r;q)_n}{(q,b_1,\dots,b_s;q)_n}\left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} z^n$$

Here we also use the q-Gamma function

$$\Gamma_q(z) = (1-q)^{1-z}\frac{(q;q)_\infty}{(q^z;q)_\infty},$$

the q-integer $[n]_q = \frac{1-q^n}{1-q}$ and the q-factorial $[n]_q! = \prod_{k=1}^n [k]_q$.

Definition (hyperpfaffian)

A hyperpfaffian is is a generalization of a Pfaffian, and first defined by [Barvinok(1995)]. Here we adopt the dedinition by [Matsumoto(2008)], which is a special case of the definition by Barvinok. Let

$$\mathfrak{E}_{2n} = \left\{\begin{pmatrix}1 & 2 & \cdots & 2n\\ \sigma_1 & \sigma_2 & \cdots & \sigma_{2n}\end{pmatrix}\in\mathfrak{S}_{2n}\,\middle|\,\sigma_{2i-1}<\sigma_{2i}\text{ for } i=1,\dots,n\right\}.$$

Let m and n be postive integers, and let $B=(B(i_1,\dots,i_{2m}))_{1\le i_1,\dots,i_{2m}\le 2n}$ be an array which satisfies

$$B(i_{\tau_1(1)},i_{\tau_1(2)},\dots,i_{\tau_m(2m-1)},i_{\tau_m(2m)}) = \operatorname{sgn}(\tau_1)\cdots\operatorname{sgn}(\tau_m)B(i_1,\dots,i_{2m})$$

for all $(\tau_1,\dots,\tau_m)\in(\mathfrak{S}_2)^m$. The hyperpfaffian $\operatorname{Pf}^{[2m]}(B)$ of B is defined by

$$\operatorname{Pf}^{[2m]}(B) = \frac{1}{n!}\sum_{\sigma_1,\dots,\sigma_m\in\mathfrak{E}_{2n}}\operatorname{sgn}(\sigma_1\cdots\sigma_m)\prod_{i=1}^n B(\sigma_1(2i-1),\sigma_1(2i),\cdots,\sigma_m(2i-1),\sigma_m(2i)).$$

If $m=1$, then $\operatorname{Pf}^{[2]}(B)$ is the ordinary Pfaffian of B, which is denoted by $\operatorname{Pf}(B)$.

Let $\{\xi_i\}_{i\ge}$ be anti-commutative symbols, i.e. $\xi_j\xi_i = -\xi_i\xi_j$. If we put

$$\zeta = \sum_{1\le k_1<k_2\le 2n}\cdots\sum_{1\le k_{2m-1}<k_{2m}\le 2n} B(k_1,\dots,k_{2m})\,\xi_{k_1}\xi_{k_2}\otimes\cdots\otimes\xi_{k_{2m-1}}\xi_{k_{2m}}$$

then we have $\zeta^n = n!\operatorname{Pf}^{[2m]}(B)\,(\xi_1\cdots\xi_{2n})^{\otimes m}$.

When $I=\{i_1,\dots,i_r\}$ is a row index set, $J=\{j_1,\dots,j_r\}$ is a column idenx set, let $A^I_J = A^{i_1,\dots,i_r}_{j_1,\dots,j_r}$ denote the $r\times r$minor of A obtained by choosing the rows in I and the columns in J. If $I=\{1,\dots,m\}$ where A has m rows, we write A_J for A^I_J.

Minor Summation Formula ([Ishikawa and Wakayama(1995)])

Let n,N be positive integers such that $2n\le N$. Let $H=(h_{i,j})_{1\le i\le 2n,1\le j\le N}$ be an $2n\times N$ rectangular matrix, and let $A=(\alpha_{i,j})_{1\le i,j\le N}$ be a skew symmetric matrix of size N. Then we have

$$\sum_{\substack{I\subseteq[N]\\ \sharp I=2n}}\operatorname{Pf}(A^I_I)\det(H_I) = \operatorname{Pf}(Q),$$

where the skew symmetric matrix Q is defined by $Q=(Q_{i,j})=HAH^{\mathrm{T}}$ whose entries may be written in the form

$$Q_{i,j} = \sum_{1\le k<l\le N}\alpha_{k,l}\det(H^{i,j}_{k,l}), \qquad (1\le i,j\le 2n).$$

Minor Summation Formula (Hyperpfaffain version [Matsumoto(2008)])

Let m, n and N be positive integers such that $2n\le N$. Let $H(s)=(h_{ij}(s))_{1\le i\le 2n,1\le j\le N}$ be $2n\times N$ rectangular matrices for $1\le s\le m$, and let $A=(\alpha_{i,j})_{1\le i,j\le N}$ be a skew symmetric matrix of size N. Then we have

$$\operatorname{Pf}^{[2m]}(Q) = \begin{cases}\sum_{\substack{I\subseteq[N]\\ \#I=2n}}\operatorname{Pf}(A^I_I)\prod_{s=1}^m\det\left(H(s)^{[2n]}_I\right) & \text{if } m \text{ is odd},\\ \sum_{\substack{I\subseteq[N]\\ \#I=2n}}\operatorname{Hf}(A_I)\prod_{s=1}^m\det\left(H(s)^{[2n]}_I\right) & \text{if } m \text{ is even}.\end{cases}$$

where the array $Q=(Q_{i_1,\dots,i_{2m}})_{1\le i_1,\dots,i_{2m}\le 2n}$ is defined by

$$Q_{i_1,\dots,i_{2m}} = \sum_{1\le k<l\le N} a_{k,l}\prod_{s=1}^m\det(H(s)^{i_{2s-1},i_{2s}}_{k,l}).$$

Proposition

Let $\{\alpha_k\}_{k\ge1}$ be any sequence, and let n be a positive integer. Let $B=(b_{i,j})_{i,j\ge1}$ be the skew-symmetric matrix defined by

$$b_{i,j} = \begin{cases}\alpha_i & \text{if } j=i+1 \text{ for } i\ge1,\\ -\alpha_j & \text{if } i=j+1 \text{ for } j\ge1,\\ 0 & \text{otherwise.}\end{cases}$$

If $I=(i_1,\dots,i_{2n})$ is an index set such that $1\le i_1<\cdots<i_{2n}$, then

$$\operatorname{Pf}\left(B^I_I\right) = \begin{cases}\prod_{k=1}^n\alpha_{i_{2k-1}} & \text{if } i_{2k}=i_{2k-1}+1 \text{ for } k=1,\dots,n,\\ 0 & \text{otherwise.}\end{cases}$$

De Bruijn's formula (hyperpfaffian version)

Let m and n be positive integers. Let $\phi_{s,i}(x)$ and $\psi_{s,i}(x)$ be functions on $[0,a]$ for $1\le i\le 2n$, $1\le s\le m$. Then we have

$$\int\cdots\int_{0\le x_1<\cdots<x_n\le a}\prod_{s=1}^m\det\left(\phi_{s,i}(x_j)|\psi_{s,i}(x_j)\right)\,\omega(d_qx)$$
$$= \operatorname{Pf}^{[2m]}\left(Q_{i_1,\cdots,i_{2m}}\right)_{1\le i_1,\cdots,i_{2m}\le 2n},$$

where

$$Q_{i_1,\cdots,i_{2m}} = \int_0^a\prod_{s=1}^m\left\{\phi_{s,i_{2s-1}}(x)\psi_{s,i_{2s}}(x) - \phi_{s,i_{2s}}(x)\psi_{s,i_{2s-1}}(x)\right\}\,\omega(d_qx)$$

for $1\le i_1,\dots,i_{2m}\le 2n$.

Corollary

Let $\omega(d_qx) = w(x)d_qx$ be a measure on $[0,a]$, and let $\mu_i = \int_0^a x^i\,\omega(d_qx)$ be the ith moment of ω. Then we have

$$\operatorname{Pf}\left((q^{i-1}-q^{j-1})\mu_{i+j+r-2}\right)_{1\le i<j\le 2n}$$
$$= \frac{q^{\binom{n}{2}}(1-q)^n}{n!}\int_{[0,a]^n}\prod_i x_i^{r+1}\prod_{i<j}(x_i-x_j)^2\prod_{i<j}(qx_i-x_j)(x_i-qx_j)\,\omega(d_qx).$$

Corollary

Let $\psi(dx) = \psi'(x)dx$ be a measure on an interval $[0,a]$, and let $\mu_i = \int_0^a x^i\,\psi(dx)$ denote the ith moment. Then we have

$$\operatorname{Pf}\left((j-i)\mu_{i+j+r-2}\right)_{1\le i<j\le 2n} = \frac{1}{n!}\int_{[0,a]^n}\prod_i x_i^{r+1}\prod_{i<j}(x_i-x_j)^4\,\psi(dx).$$

Corollary

Let $\psi(dx) = \psi'(x)dx$ be a measure on an interval $[0,a]$, and let $\mu_i = \int_0^a x^i\,\psi(dx)$ denote the ith moment. Then we have

$$\operatorname{Pf}^{[2m]}\left(\prod_{s=1}^m(i_{2s}-i_{2s-1})\cdot\mu_{i_1+\cdots+i_{2m}+r}\right)_{0\le i<j\le 2n-1}$$
$$= \frac{1}{n!}\int_{[a,b]^n}\prod_i x_i^{r+m}\prod_{i<j}(x_i-x_j)^{4m}\,\psi(dx).$$

Theorem

For integers $n\ge1$ and $r\ge0$, we have

$$\operatorname{Pf}\left((q^{i-1}-q^{j-1})\frac{(aq;q)_{i+j+r-2}}{(abq^2;q)_{i+j+r-2}}\right)_{1\le i,j\le 2n}$$
$$= a^{n(n-1)}q^{n(n-1)(4n+1)/3+n(n-1)r}\prod_{k=1}^{n-1}(bq;q)_{2k}\prod_{k=1}^n\frac{(q;q)_{2k-1}(aq;q)_{2k+r-1}}{(abq^2;q)_{2(k+n)+r-3}}.$$

Let ω be the measure on $[0,1]$ defined by

$$\int_0^1 f(x)\,\omega(d_qx) = \frac{(aq;q)_\infty}{(abq^2;q)_\infty}\sum_{k=0}^\infty\frac{(bq;q)_k}{(q;q)_k}(aq)^k f\left(q^k\right)$$

which implies

$$w(x) = \frac{1}{1-q}\cdot\frac{(aq,bq;q)_\infty}{(abq^2,q;q)_\infty}\cdot\frac{(qx;q)_\infty}{(bqx;q)_\infty}x^{\alpha+1},$$

where $a=q^\alpha$. The nth moment is given by

$$\mu_n = \int_0^1 x^n\,\omega(d_qx) = \frac{(aq;q)_n}{(abq^2;q)_n} \quad (n=0,1,2,\dots),$$

which is the moment of the Little q-Jacobi polynomials [Gasper and Rahman(2004)]

$$p_n(x;a,b;q) = \frac{(aq;q)_n}{(abq^{n+1};q)_n}(-1)^n q^{\binom{n}{2}}{}_2\phi_1\left[\begin{matrix}q^{-n},abq^{n+1}\\ aq\end{matrix};q,xq\right].$$

The q-gamma function is defined on $\mathbb{C}\setminus\mathbb{Z}_{<0}$ by

$$\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty}(1-q)^{1-a}.$$

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