Schur Function Identities and Hook Length Posets

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ABSTRACT. In this paper we find new classes of posets which generalize the d-complete posets. In fact the d-complete posets are classified into 15 irreducible classes in the paper “Dynkin diagram classification of minuscule Bruhat lattices and of d-complete posets” (J. Algebraic Combin. 9 (1999), 61 – 94) by R. A. Proctor. Here we present six new classes of hook length property which generalize the 15 irreducible classes. Our method to prove the hook length property is based on R. P. Stanley’s (P,v)-partitions and Schur function identities.

INTRODUCTION

In [2] R. A. Proctor defined d-complete posets, which include shapes, shifted shapes and trees, by certain local structural Schur function $I_{\beta}^{\gamma}$ by R. A. Proctor. Here we present six new classes of hook length property which generalize the 15 irreducible classes. Our method to prove the hook length property is based on R. P. Stanley’s (P,v)-partitions and Schur function identities.

Schur Function Identities

In this section we state eight Candy type identities of the Schur functions, which will be applied in the following sections. The Schur function $s_{\lambda}(x_1, x_2, \ldots, x_n)$ of variables $x_1, x_2, \ldots, x_n$ with respect to a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is defined to be

$$s_{\lambda}(x_1, x_2, \ldots, x_n) = \frac{\det(x_i^{\lambda_j-1} + 1_{j \leq i})}{\det(x_i^{\lambda_j})}.$$

For a positive integer m, we write $X_m = (x_1, x_2, \ldots, x_m)$, $Y_m = (y_1, y_2, \ldots, y_m)$ and $Z_m = (z_1, z_2, \ldots, z_m)$ in short. Let $\mathcal{A}$ denote the set of all partitions. If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition and a and b are positive integers such that $a \leq b$, then we write $\lambda_{(a,b)}$ in short, for the partition $(\lambda_1, \lambda_2, \ldots)$. If $\lambda = (x_1, x_2, \ldots, x_m)$ is an m-tuple of variables, then we use the notation $X_{\lambda_m} = \prod x_i^{\lambda_i}$ for brevity. We prove the following variants of the Candy identity.

Theorem 2.1. Let m be a positive integer. (i) If m = 2, then we have

$$\sum_{\lambda \in \mathcal{A}} \chi_{\lambda}(\mathbf{w}) s_{\lambda}(X_2) s_{\lambda}(Y_2) = I_{\lambda}^{\gamma}(Y_2) - I_{\lambda}^{\gamma}(X_2).$$

(ii) If m ≥ 2 and r = 1 or 2, then we have

$$\sum_{\lambda \in \mathcal{A}} \chi_{\lambda}(\mathbf{w}) s_{\lambda}(X_m) s_{\lambda}(Y_m) = I_{\lambda}^{\gamma}(Y_m) - \prod_{i=1}^{r-1} I_{\lambda-r}^{\gamma}(X_{r-i}) I_{\lambda}^{\gamma}(X_r).$$

(iii) If m ≥ 3 and r = 1 or 2, then we have

$$\sum_{\lambda \in \mathcal{A}} \chi_{\lambda}(\mathbf{w}) s_{\lambda}(X_m) s_{\lambda}(Y_m) = \prod_{i=1}^{r-1} I_{\lambda-r}^{\gamma}(X_{r-i}) I_{\lambda}^{\gamma}(X_r).$$

(iv) If m ≥ 3 and r = 1 or 2, then we have

$$\sum_{\lambda \in \mathcal{A}} \chi_{\lambda}(\mathbf{w}) s_{\lambda}(X_m) s_{\lambda}(Y_m) = \prod_{i=1}^{r-1} I_{\lambda-r}^{\gamma}(X_{r-i}) I_{\lambda}^{\gamma}(X_r).$$

We call these new classes of hook length posets, which we call “leaf posets.”

Hook Length Posets

The aim of this section is to define new classes of hook length posets which include any irreducible d-complete poset. We call these classes basic leaf posets. Let $B_{\lambda}$ be a partially ordered set (poset). A $\mathcal{P}$-partition is a map $\varphi$ from $B_{\lambda}$ to $\{1, 2, \ldots, m\}$ satisfying that $\varphi(x) \geq \varphi(y)$ if $x < y$ in $B_{\lambda}$, i.e. $\varphi$ is order reversing. We denote the set of all $\mathcal{P}$-partitions by $\mathcal{P}(B_{\lambda})$. We say that $B_{\lambda}$ is a hook length poset if there exists a map $\varphi$ from $B_{\lambda}$ to $\{1, 2, \ldots, m\}$ satisfying

$$\sum_{\varphi(x) < \varphi(y)} I_{\varphi(y)}^{\varphi(x)}(x) = \sum_{\varphi(x) = \varphi(y)} I_{\varphi(y)}^{\varphi(x)}(x)$$

It is well known that shapes, shifted shapes and trees are hook length posets. From now on, we denote the set of the strictly decreasing sequence of nonnegative integers by $\mathcal{J}$. Basic leaf posets are defined as follows.

Definition 3.1. (i) Let $m \geq 2$ be an integer, and let $\mathcal{J} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be an m-tuple of variables, and let $\alpha, \beta, \gamma, \delta$ be elements of $\mathcal{J}$. Let $\delta$ and $\beta$ be nonnegative integers which satisfy $f \geq 2 \delta + g \geq 0$. Then, a $\mathcal{J}$-ginkgo $\mathcal{G}(\alpha, \beta, \gamma, \delta)$ is a poset defined by the diagram in Figure 1 in the map $\varphi$ denotes the chain of length $\delta$.

(iii) Let $m \geq 2$ be an integer, and let $\mathcal{J} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be an m-tuple of variables, and let $\alpha, \beta, \gamma, \delta$ be elements of $\mathcal{J}$. Let $\delta$ and $\beta$ be nonnegative integers which satisfy $f \geq 2 \delta + g \geq 0$. Then, a $\mathcal{J}$-bamboo $\mathcal{B}(\alpha, \beta, \gamma, \delta)$ is a poset defined by the diagram in Figure 2.

(iii) Let $m \geq 2$ be a positive integer, and let $\mathcal{J} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be an m-tuple of variables, and let $\alpha, \beta, \gamma, \delta$ be elements of $\mathcal{J}$. Let $\delta$ and $\beta$ be nonnegative integers which satisfy $f \geq 2 \delta + g \geq 0$. Then, a $\mathcal{J}$-mistletoe $\mathcal{M}(\alpha, \beta, \gamma, \delta)$ is a poset defined by the diagram in Figure 3.

(iii) Let $m \geq 2$ be a positive integer, and let $\mathcal{J} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be an m-tuple of variables, and let $\alpha, \beta, \gamma, \delta$ be elements of $\mathcal{J}$. Let $\delta$ and $\beta$ be nonnegative integers which satisfy $f \geq 2 \delta + g \geq 0$. Then, a $\mathcal{J}$-ivy $\mathcal{I}(\alpha, \beta, \gamma, \delta)$ is a poset defined by the diagram in Figure 4.

(iii) Let $m \geq 2$ be a positive integer, and let $\mathcal{J} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be an m-tuple of variables, and let $\alpha, \beta, \gamma, \delta$ be elements of $\mathcal{J}$. Let $\delta$ and $\beta$ be nonnegative integers which satisfy $f \geq 2 \delta + g \geq 0$. Then, a $\mathcal{J}$-chrysanths $\mathcal{C}(\alpha, \beta, \gamma, \delta)$ is a poset defined by the diagram in Figure 5.

(iii) Let $m \geq 2$ be a positive integer, and let $\mathcal{J} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be an m-tuple of variables, and let $\alpha, \beta, \gamma, \delta$ be elements of $\mathcal{J}$. Let $\delta$ and $\beta$ be nonnegative integers which satisfy $f \geq 2 \delta + g \geq 0$. Then, a $\mathcal{J}$-firs $\mathcal{F}(\alpha, \beta, \gamma, \delta)$ is a poset defined by the diagram in Figure 6.

Corollary 3.2. Any basic leaf poset is a hook length poset. In particular, any d-complete poset is a hook length poset since it can be realized as a leaf poset.

A general leaf poset defined from the basic ones by using an operation called “joint sum,” which is a similar operation called “slant sum” introduced in [2] in order to combine two irreducible d-complete posets to generate a general d-complete poset. We conclude that any leaf poset is a hook length poset.

References
