Minor summation formula and a proof of Stanley's open problem

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References

- M. Ishikawa, "Minor summation formula and a proof of Stanley's open problem", `arXiv:math.CO/0408204`.
- R. P. Stanley, "Open problem", International Conference on Formal Power Series and Algebraic Combinatorics (Vadstena 2003), June 23 - 27, 2003, available from http://www-math.mit.edu/r̃stan/trans.html.

Partitions

A partition is any (finite or infinite) sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$$

of non-negative integers in weakly decreasing order:

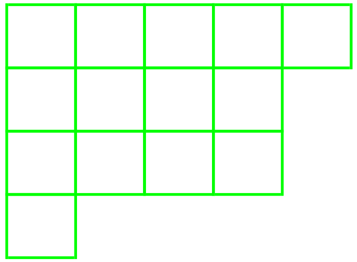
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \ldots$$

and containing only finitely many non-zero terms. The non-zero terms λ_i are called the parts of λ. The number of parts is the length of λ, denoted by $\ell(\lambda)$; and the sum of the parts is the weight of λ, denoted by $|\lambda|$.

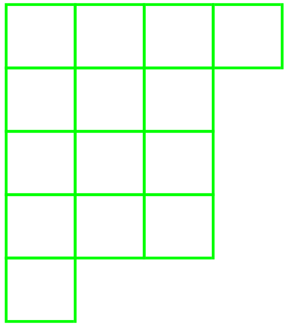
Let $\mathbb{P}$ denote the set of positive integers. Consider the elements of $\mathbb{P}^2$, regarded as the lattice points of $\mathbb{R}^2$ in the positive quadrant. The (Ferrers) diagram of a partition λ may be formally defined as the set of lattice points such that $1 \leq j \leq \lambda_i$. It is convenient to replace the points by squares. The conjuate of a partition λ is the partition λ' whose diagram is the transpose of the diagram of λ.

Example

$\boldsymbol{\lambda} = (5441)$ is a partition with length 4 and weight 14.



Its conjugate is $\boldsymbol{\lambda} = (43331)$ and its Ferrers diagram is as follow:



The Schur functions

For $X = (x_1, \ldots, x_n)$ and a partition λ such that $\ell(\lambda) \le n$,

$$s_\lambda(X) = \frac{\det(x_i^{\lambda_j+n-j})_{1\le i,j\le n}}{\det(x_i^{n-j})_{1\le i,j\le n}}.$$

Tableaux

Given a partition λ, A tableaux T of shape λ is a filling of the diagram with numbers whereas the numbers must strictly increase down each column and weakly from left to right along each row.

Schur functions

The Schur function $s_\lambda(x)$ is

$$s_\lambda(X) = \sum_T X^T,$$

where the sum runs over all tableaux of shape λ. Here $X^T = x_1^{\sharp 1\text{s in } T} x_2^{\sharp 2\text{s in } T} \cdots$

Example

A Tableau T of shape (5441).

1	1	1	2	2
2	2	3	4	
3	3	4	5	
5				

The weight of T is $x_1^3 x_2^4 x_3^3 x_4^2 x_5^2$.

Example

When $\lambda = (2,2)$ and $X = (x_1, x_2, x_3, x_4)$,

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 2&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 4&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 2&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 4&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 3&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 4&4\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 3&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 4&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 2&3\\\hline 4&4\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 3&3\\\hline 4&4\\\hline\end{array}$$

$$\begin{aligned} s_\lambda(X) = {} & x_1^2x_2^2 + x_1^2x_3^2 + x_1^2x_4^2 + x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2 + 2x_1x_2x_3x_4 \\ & + x_1^2x_2x_3 + x_1^2x_2x_4 + x_1^2x_3x_4 + x_2^2x_1x_3 + x_2^2x_1x_4 + x_2^2x_3x_4 \\ & + x_3^2x_1x_2 + x_3^2x_1x_4 + x_3^2x_2x_4 + x_4^2x_1x_2 + x_4^2x_1x_3 + x_4^2x_2x_3 \end{aligned}$$

Power Sum Symmetric Functions

Let r denote a positive integer.

$$p_r(X) = x_1^r + x_2^r + \cdots + x_n$$

is called the rth power sum symmetric functions.

$$p_1(X) = x_1 + x_2 + \cdots + x_n$$

$$p_2(X) = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$p_3(X) = x_1^3 + x_2^3 + \cdots + x_n^3$$

The ring of symmetric functions

The ring Λ of symmetric functions in countably many variables x_1, x_2, $\dots$ is defined by the inverse limit. (See the detains in Macdonald's book I, 2.)

Here we use the convention that $f(x)$ stands for a symmetric function in countably many variables $x = (x_1, x_2, \dots)$, whereas $f(X)$ stands for a symmetric function in finitely many variables $X = (x_1, \dots, x_n)$.

Stanley's weight

Given a partition λ, define $\omega(\lambda)$ by

$$\omega(\lambda) = a^{\sum_{i\geq 1}\lceil \lambda_{2i-1}/2\rceil} b^{\sum_{i\geq 1}\lfloor \lambda_{2i-1}/2\rfloor} c^{\sum_{i\geq 1}\lceil \lambda_{2i}/2\rceil} d^{\sum_{i\geq 1}\lfloor \lambda_{2i}/2\rfloor},$$

where a, b, c and d are indeterminates, and $\lceil x\rceil$ (resp. $\lfloor x\rfloor$) stands for the smallest (resp. largest) integer greater (resp. less) than or equal to x for a given real number x. For example, if $\lambda = (5, 4, 4, 1)$ then $\omega(\lambda)$ is the product of the entries in the following diagram for λ, which is equal to $a^5b^4c^3d^2$.

a	b	a	b	a
c	d	c	d	
a	b	a	b	
c				

Stanley's open problem

In FPSAC'03 R.P. Stanley gave the following problem in the open problem session:

Theorem

Let

$$z = \sum_{\lambda} \omega(\lambda) s_{\lambda},$$

where the sum runs over all partitions λ.

Then we have

$$\log z - \sum_{n\geq 1} \frac{1}{2n} a^n (b^n - c^n) p_{2n} - \sum_{n\geq 1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \ldots]].$$

A simple version

Let

$$y = \sum_{\substack{\lambda \\ \lambda,\lambda' \text{ even}}} s_\lambda(x).$$

Here the sum runs over all partitions λ such that λ and λ' are even partitions (i.e. with all parts even).

Then we have

$$\log y - \sum_{n \geq 1} \frac{1}{4n} p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \ldots]].$$

Strategy of the proof

1. Step1. Express $\omega(\lambda)$ and z by a Pfaffian.

   Use our minor summation formula of Pfaffians.

2. Step2. Express z by a determinant.

   A Homogenious version of Okada's gereralization of Schur's Pfaffian.

3. Step3. Show that

$$\log z - \sum_{n\geq 1} \frac{1}{2n} a^n (b^n - c^n) p_{2n} - \sum_{n\geq 1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \dots]].$$

   Use Stembridge's criterion.

The goal of the proof

Put

$$w = \log z - \sum_{n \geq 1} \frac{1}{2n} a^n (b^n - c^n) p_{2n} - \sum_{n \geq 1} \frac{1}{4n} a^n b^n c^n d^n p_{2n}^2$$

and use the following Stembridge's criterion to w.

Proposition (Stembridge)

Let $f(x_1, x_2, \dots)$ be a symmetric function with infinite variables. Then

$$f \in \mathbb{Q}[p_\lambda : \text{ all parts } \lambda_i > 0 \text{ are odd}]$$

if and only if

$$f(t, -t, x_1, x_2, \dots) = f(x_1, x_2, \dots).$$

Pfaffians

Assume we are given a $2n$ by $2n$ skew-symmetric matrix

$$A = (a_{ij})_{1 \le i,j \le 2n},$$

(i.e. $a_{ji} = -a_{ij}$), whose entries a_{ij} are in a commutative ring.

The Pfaffian of A is, by definition,

$$\mathrm{Pf}(A) = \frac{1}{n!} \sum \epsilon(\sigma_1, \sigma_2, \dots, \sigma_{2n-1}, \sigma_{2n}) a_{\sigma_1 \sigma_2} \cdots a_{\sigma_{2n-1} \sigma_{2n}}.$$

where the summation is over all partitions $\{\{\sigma_1, \sigma_2\}_<, \dots, \{\sigma_{2n-1}, \sigma_{2n}\}_<\}$ of $[2n]$ into 2-elements blocks, and where $\epsilon(\sigma_1, \sigma_2, \dots, \sigma_{2n-1}, \sigma_{2n})$ denotes the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & 2n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{2n} \end{pmatrix}.$$

Example

When $n = 2$,

$$\mathrm{Pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{21} & 0 & a_{23} & a_{24} \\ -a_{31} & -a_{32} & 0 & a_{34} \\ -a_{41} & -a_{42} & -a_{43} & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

The aim of Step1

Can we express z by a Pfaffian?

Theorem A

Let n be a positive integer. Let

$$z_n = \sum_{\ell(\lambda) \le 2n} \omega(\lambda) s_\lambda(X_{2n})$$

be the sum restricted to $2n$ variables. Then we have

$$z_n = \frac{1}{\prod_{1 \le i < j \le 2n} (x_i - x_j)} (abcd)^{-\binom{n}{2}} \mathrm{Pf}\, (p_{ij})_{1 \le i < j \le 2n},$$

where p_{ij} is defined by

$$\frac{\begin{vmatrix} x_i + a x_i^2 & 1 - a(b+c)x_i - abcx_i^3 \\ x_j + a x_j^2 & 1 - a(b+c)x_j - abcx_j^3 \end{vmatrix}}{(1 - abx_i^2)(1 - abx_j^2)(1 - abcdx_i^2 x_j^2)}.$$

The key to the proof of this theorem

Can we express the weight $\omega(\lambda)$ by a Pfaffian?

Notation

Let m, n and r be integers such that $r \leq m, n$. Let A be an m by n matrix. For any index sets

$$I = \{i_1, \ldots, i_r\}_< \subseteq [m],$$
$$J = \{j_1, \ldots, j_r\}_< \subseteq [n],$$

let $\Delta_J^I(A)$ denote the submatrix obtained by selecting the rows indexed by I and the columns indexed by J. If $r = m$ and $I = [m]$, we simply write $\Delta_J(A)$ for $\Delta_J^{[m]}(A)$. Similarly, if $r = n$ and $J = [n]$, we write $\Delta^I(A)$ for $\Delta_{[n]}^I(A)$. For any finite set S and a non-negative integer r, let $\binom{S}{r}$ denote the set of all r-element subsets of S.

Notation

Let n be a non-negative integer.

Let $\lambda = (\lambda_1, \dots, \lambda_{2n})$ be a partition such that $\ell(\lambda) \leq 2n$.

Put

$$l = (l_1, \dots, l_{2n}) = (\lambda_1 + 2n - 1, \dots, \lambda_{2n}) = \lambda + \delta_{2n}.$$

where $\delta_{2n} = (2n-1, 2n-2, \dots, 0)$.

Let

$$I(\lambda) = \{l_1, \dots, l_{2n}\}.$$

We regard this set as a set of row/column indices.

Example

If $n = 3$ and $\lambda = (5, 4, 4, 1, 0, 0)$, then

$$l = \lambda + \delta_6 = (10, 8, 7, 3, 1, 0),$$

and

$$I(\lambda) = \{0, 1, 3, 7, 8, 10\}.$$

Theorem

Define a skew-symmetric array $A = (\alpha_{ij})_{0 \le i,j}$ by

$$\alpha_{ij} = a^{\lceil (j-1)/2 \rceil} b^{\lfloor (j-1)/2 \rfloor} c^{\lceil i/2 \rceil} d^{\lfloor i/2 \rfloor}$$

for $i < j$.

Then we have

$$\mathrm{Pf}\left[A_{I(\lambda)}^{I(\lambda)}\right] = (abcd)^{\binom{n}{2}} \omega(\lambda).$$

Lemma

Let x_i and y_j be indeterminates, and let n is a non-negative integer.

Then

$$\mathrm{Pf}\left[x_i y_j\right]_{1 \le i < j \le 2n} = \prod_{i=1}^{n} x_{2i-1} \prod_{i=1}^{n} y_{2i}.$$

Example

$A = (\alpha_{ij})_{0 \le i,j}$:

$$\begin{bmatrix} 0 & 1 & a & ab & a^2b & a^2b^2 & \dots \\ -1 & 0 & ac & abc & a^2bc & a^2b^2c & \dots \\ -a & -ac & 0 & abcd & a^2bcd & a^2b^2cd & \dots \\ -ab & -abc & -abcd & 0 & a^2bc^2d & a^2b^2c^2d & \dots \\ -a^2b & -a^2bc & -a^2bcd & -a^2bc^2d & 0 & a^2b^2c^2d^2 & \dots \\ -a^2b^2 & -a^2b^2c & -a^2b^2cd & -a^2b^2c^2d & -a^2b^2c^2d^2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example

If $n = 3$. and $\lambda = (5, 4, 4, 1, 0, 0)$, then

$$I(\lambda) = \{0, 1, 3, 7, 8, 10\}.$$

$A_{I(\lambda)}^{I(\lambda)}$:

$$\begin{bmatrix} 0 & 1 & ab & a^3b^3 & a^4b^3 & a^5b^4 \\ -1 & 0 & abc & a^3b^3c & a^4b^3c & a^5b^4c \\ -ab & -abc & 0 & a^3b^3c^2d & a^4b^3c^2d & a^5b^4c^2d \\ -a^3b^3 & -a^3b^3c & -a^3b^3c^2d & 0 & a^4b^3c^4d^3 & a^5b^4c^4d^3 \\ -a^4b^3 & -a^4b^3c & -a^4b^3c^2d & -a^4b^3c^4d^3 & 0 & a^5b^4c^4d^4 \\ -a^5b^4 & -a^5b^4c & -a^5b^4c^2d & -a^5b^4c^4d^3 & -a^5b^4c^4d^4 & 0 \end{bmatrix}$$

$$\mathrm{Pf}\left(A_{I(\lambda)}^{I(\lambda)}\right) = a^8b^7c^6d^5 = (abcd)^3\omega(\lambda)$$

Schur functions

Let $X = (x_1, \ldots, x_{2n})$ and let $T = (x_i^{j-1})_{1 \le i \le 2n, j \ge 1}$, i.e.

$$T = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \ldots \\ 1 & x_2 & x_2^2 & x_2^3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & x_{2n} & x_{2n}^2 & x_{2n}^3 & \ldots \end{bmatrix}$$

Then we have

$$s_\lambda(X) = \frac{\det\left(\Delta_{I(\lambda)}(T)\right)_{1 \le i,j \le 2n}}{\det(x_i^{j-1})_{1 \le i,j \le 2n}}.$$

Example

If $n = 3$. and $\lambda = (5, 4, 4, 1, 0, 0)$, then

$$I(\lambda) = \{0, 1, 3, 7, 8, 10\},$$

and

$$s_\lambda(X) = \frac{\begin{vmatrix} 1 & x_1 & x_1^3 & x_1^7 & x_1^8 & x_1^{10} \\ 1 & x_2 & x_2^3 & x_2^7 & x_2^8 & x_2^{10} \\ 1 & x_3 & x_3^3 & x_3^7 & x_3^8 & x_3^{10} \\ 1 & x_4 & x_4^3 & x_4^7 & x_4^8 & x_4^{10} \\ 1 & x_5 & x_5^3 & x_5^7 & x_5^8 & x_5^{10} \\ 1 & x_6 & x_6^3 & x_6^7 & x_6^8 & x_6^{10} \end{vmatrix}}{\prod_{1 \le i < j \le 6} (x_j - x_i)}$$

Theorem (Minor summation formula)

Let n and N be non-negative integers such that $2n \leq N$. Let $T = (t_{ij})_{1 \leq i \leq 2n, 1 \leq j \leq N}$ be a $2n$ by N rectangular matrix, and let $A = (a_{ij})_{1 \leq i,j \leq N}$ be a skew-symmetric matrix of size N. Then

$$\sum_{I \in \binom{[N]}{2n}} \mathrm{Pf}\left(\Delta_I^I(A)\right) \det\left(\Delta_I(T)\right) = \mathrm{Pf}\left(T A\, {}^t T\right).$$

If we put $Q = (Q_{ij})_{1 \leq i,j \leq 2n} = T A\, {}^t T$, then its entries are given by

$$Q_{ij} = \sum_{1 \leq k < l \leq N} a_{kl} \det\left(\Delta_{kl}^{ij}(T)\right),$$

$(1 \leq i, j \leq 2n)$. Here we write $\Delta_{kl}^{ij}(T)$ for

$$\Delta_{\{kl\}}^{\{ij\}}(T) = \begin{vmatrix} t_{ik} & t_{il} \\ t_{jk} & t_{jl} \end{vmatrix}.$$

The idea of the proof of Theorem A

- The Schur function $s_\lambda(X_{2n})$ is a quotient of determinants. (The denominator is the Vandermonde determinant.)
- The weight $\omega(\lambda)$ is a Pfaffian.
- Take the product of the Pfaffian and the determinant, and take the sum over all columns.

The aim of Step2

Can we express the Pfaffian by a determinant?

Generalized Vandermonde determinants

Let $X = (x_1, \cdots, x_n)$ and $A = (a_1, \cdots, a_n)$ be two vectors of variables of length n. For nonnegative integers p and q with $p + q = n$, we define a generalized Vandermonde matrix $V^{p,q}(X; A)$ to be the $n \times n$ matrix with ith row

$$(a_i x_i^{p-1}, a_i x_i^{p-2}, \cdots, a_i, x_i^{q-1}, x_i^{q-2}, \cdots, 1).$$

We introduce another generalized Vandermonde matrix $W^n(X; A)$ as the $n \times n$ matrix with ith row

$$(1 + a_i x_i^{n-1}, x_i + a_i x_i^{n-2}, \cdots, x_i^{n-1} + a_i).$$

If $p = 0$, then $V^{0,n}(X; A) = \left(x_i^{n-j} \right)_{1 \leq i,j \leq n}$ and the determinant $\det V^{0,n}(X; A) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$ is the ordinary Vandermonde determinant.

Example

When $p = 4$ and $q = 3$, $V^{4,3}(X; A)$ is

$$\begin{bmatrix} a_1x_1^3 & a_1x_1^2 & a_1x_1 & a_1 & x_1^2 & x_1 & 1 \\ a_2x_2^3 & a_2x_2^2 & a_2x_2 & a_2 & x_2^2 & x_2 & 1 \\ a_3x_3^3 & a_3x_3^2 & a_3x_3 & a_3 & x_3^2 & x_3 & 1 \\ a_4x_4^3 & a_4x_4^2 & a_4x_4 & a_4 & x_4^2 & x_4 & 1 \\ a_5x_5^3 & a_5x_5^2 & a_5x_5 & a_5 & x_5^2 & x_5 & 1 \\ a_6x_6^3 & a_6x_6^2 & a_6x_6 & a_6 & x_6^2 & x_6 & 1 \\ a_7x_7^3 & a_7x_7^2 & a_7x_7 & a_7 & x_7^2 & x_7 & 1 \end{bmatrix}.$$

A homogeneous version

Let $X = (x_1, \cdots, x_n)$, $Y = (y_1, \cdots, y_n)$, $A = (a_1, \cdots, a_n)$ and $B = (b_1, \cdots, b_n)$ be four vectors of variables of length n. For nonnegative integers p and q with $p + q = n$, Define a generalized Vandermonde matrix $U^{p,q}(X, Y; A, B)$ by the $n \times n$ matrix with ith row

$$(a_i x_i^{p-1}, a_i x_i^{p-2} y_i, \cdots, a_i y_i^{p-1}, b_i x_i^{q-1}, b_i x_i^{p-2} y_i, \cdots, b_i y_i^{q-1}).$$

If we substitute $B = Y = \mathbf{1}$, we have

$$U^{p,q}(X, \mathbf{1}; A, \mathbf{1}) = V^{p,q}(X; A),$$

where $\mathbf{1} = (1, 1, \ldots, 1)$.

Example

When $p = q = 1$,

$$U^{1,1}(X, Y; A, B) = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}.$$

When $p = q = 2$,

$$U^{2,2}(X, Y; A, B) = \begin{bmatrix} a_1 x_1 & a_1 y_1 & b_1 x_1 & b_1 y_1 \\ a_2 x_2 & a_2 y_2 & b_2 x_2 & b_2 y_2 \\ a_3 x_3 & a_3 y_3 & b_3 x_3 & b_3 y_3 \\ a_4 x_4 & a_4 y_4 & b_4 x_4 & b_4 y_4 \end{bmatrix}.$$

When $p = q = 3$, $U^{3,3}(X, Y; A, B)$ is

$$\begin{bmatrix} a_1x_1^2 & a_1x_1y_1 & a_1y_1^2 & b_1x_1^2 & b_1x_1y_1 & b_1y_1^2 \\ a_2x_2^2 & a_2x_2y_2 & a_2y_2^2 & b_2x_2^2 & b_2x_2y_2 & b_2y_2^2 \\ a_3x_3^2 & a_3x_3y_3 & a_3y_3^2 & b_3x_3^2 & b_3x_3y_3 & b_3y_3^2 \\ a_4x_4^2 & a_4x_4y_4 & a_4y_4^2 & b_4x_4^2 & b_4x_4y_4 & b_4y_4^2 \\ a_5x_5^2 & a_5x_5y_5 & a_5y_5^2 & b_5x_5^2 & b_5x_5y_5 & b_5y_5^2 \\ a_6x_6^2 & a_6x_6y_6 & a_6y_6^2 & b_6x_6^2 & b_6x_6y_6 & b_6y_6^2 \end{bmatrix}.$$

The aim of Step2 is to prove the following theorem:

Theorem B

Let $X = (x_1, \dots, x_{2n})$ be a $2n$-tuple of variables. Then

$$z_n(X_{2n}) = (-1)^{\binom{n}{2}} \times \frac{\det U^{n,n}(X^2, \mathbf{1} + abcdX^4; X + aX^2, \mathbf{1} - a(b+c)X^2 - abcX^3)}{\prod_{i=1}^{2n}(1 - abx_i^2) \prod_{1 \le i < j \le 2n}(x_i - x_j)(1 - abcdx_i^2 x_j^2)},$$

where $X^2 = (x_1^2, \dots, x_{2n}^2)$, $\mathbf{1} + abcdX^4 = (1 + abcdx_1^4, \dots, 1 + abcdx_{2n}^4)$, $X + aX^2 = (x_1 + ax_1^2, \dots, x_{2n} + ax_{2n}^2)$ and $\mathbf{1} - a(b+c)X^2 - abcX^3 = (1 - a(b+c)x_1^2 - abcx_1^3, \dots, 1 - a(b+c)x_{2n}^2 - abcx_{2n}^3)$.

Cauchy's determinant

$$\det\left[\frac{1}{x_i+y_j}\right]_{1\le i,j\le n} = \frac{\Delta_n(X)\Delta_n(Y)}{\prod_{1\le i,j\le n}(x_i+y_j)}.$$

Schur's Pfaffian

$$\mathrm{Pf}\left[\frac{x_i-x_j}{x_i+x_j}\right]_{1\le i,j\le 2n} = \frac{\Delta_{2n}(X)}{\prod_{1\le i<j\le 2n}(x_i+x_j)}.$$

Here $\Delta_n(X)=\prod_{1\le i<j\le n}(x_i-x_j)$.

Generalizations of Cauchy's determinant and Schur's Pfaffian

First, Soichi Okada presented the following identities at the workshop on "Aspects of Combinatorial Representation Theory" (Octorber, 2003) and "2nd East Asian Conference on Algebra and Combinatorics" (November, 2003). At the point they are conjectures.

Theorem

(a) Let n be a positive integer and let p and q be nonnegative integers. For six vectors of variables

$$X = (x_1, \cdots, x_n),\ Y = (y_1, \cdots, y_n),\ A = (a_1, \cdots, a_n),$$
$$B = (b_1, \cdots, b_n),\ Z = (z_1, \cdots, z_{p+q}),\ C = (c_1, \cdots, c_{p+q}),$$

we have

$$\det \left(\frac{\det V^{p+1,q+1}(x_i, y_j, Z; a_i, b_j, C)}{y_j - x_i} \right)_{1 \le i,j \le n}$$
$$= \frac{(-1)^{n(n-1)/2}}{\prod_{i,j=1}^{n} (y_j - x_i)} \det V^{p,q}(Z;C)^{n-1}$$
$$\times \det V^{n+p,n+q}(X, Y, Z; A, B, C).$$

(b) Let n be a positive integer and let p, q, r, s be nonnegative integers. For seven vectors of variables

$$X = (x_1, \cdots, x_{2n}),\ A = (a_1, \cdots, a_{2n}),\ B = (b_1, \cdots, b_{2n}),$$
$$Z = (z_1, \cdots, z_{p+q}),\ C = (c_1, \cdots, c_{p+q}),$$
$$W = (w_1, \cdots, w_{r+s}),\ D = (d_1, \cdots, d_{r+s}),$$

we have

$$\mathrm{Pf}\left(\frac{\det V^{p+1,q+1}(x_i, x_j, Z; a_i, a_j, C) \det V^{r+1,s+1}(x_i, x_j, W; b_i, b_j, D)}{x_j - x_i} \right)_1$$
$$= \frac{1}{\prod_{1 \le i < j \le 2n}(x_j - x_i)} \det V^{p,q}(Z; C)^{n-1} \det V^{r,s}(W; D)^{n-1}$$
$$\times \det V^{n+p,n+q}(X, Z; A, C) \det V^{n+r,n+s}(X, W; B, D).$$

(c) Let n be a positive integer and let p be a nonnegative integer. For six vectors of variables

$$X = (x_1, \cdots, x_n),\ Y = (y_1, \cdots, y_n),\ A = (a_1, \cdots, a_n),$$
$$B = (b_1, \cdots, b_n),\ Z = (z_1, \cdots, z_p),\ C = (c_1, \cdots, c_p),$$

we have

$$\det\left(\frac{\det W^{p+2}(x_i, y_j, Z; a_i, b_j, C)}{(y_j - x_i)(1 - x_i y_j)}\right)_{1\le i,j\le n}$$
$$= \frac{1}{\prod_{i,j=1}^{n}(y_j - x_i)(1 - x_i y_j)} \det W^{p}(Z; C)^{n-1}$$
$$\times \det W^{2n+p}(X, Y, Z; A, B, C).$$

(d) Let n be a positive integer and let p and q be nonnegative integers. For seven vectors of variables

$$X = (x_1, \cdots, x_{2n}),\ A = (a_1, \cdots, a_{2n}),\ B = (b_1, \cdots, b_{2n}),$$
$$Z = (z_1, \cdots, z_p),\ C = (c_1, \cdots, c_p),$$
$$W = (w_1, \cdots, w_q),\ D = (d_1, \cdots, d_q),$$

we have

$$\mathrm{Pf}\left(\frac{\det W^{p+2}(x_i, x_j, Z; a_i, a_j, C) \det W^{q+2}(x_i, x_j, W; b_i, b_j, D)}{(x_j - x_i)(1 - x_i x_j)}\right)_{1 \le i < j \le 2n}$$
$$= \frac{1}{\prod_{1 \le i < j \le 2n}(x_j - x_i)(1 - x_i x_j)} \det W^p(Z; C)^{n-1} \det W^q(W; D)^{n-1}$$
$$\times \det W^{2n+p}(X, Z; A, C) \det W^{2n+q}(X, W; B, D).$$

The background of Okada's conjecture

The background of these formulae is in

S. Okada, "Enumeration of symmetry classes of alternating sign matrices and characters of classical groups", `arXiv:math.CO/0308234`.

His work is based on

G. Kuperberg, "Symmetry classes of alternating-sign matrices under one roof", Ann. of Math. (2) **156** (2002), 835-866.

The idea of the proof

The proof is given in

> M. Ishikawa, S. Okada, H. Tagawa and J. Zeng Generalizations of Cauchy's determinant and Schur's Pfaffian", `arXiv:math.CO/0411280`.

We not only proved his conjecture but also gathered more generalizations of Cauchy's determinant and Schur's Pfaffian and their applications.

Theorem (The Desnanot–Jacobi formulae)

(1) If A is a square matrix, then we have

$$\det A_1^1 \cdot \det A_2^2 - \det A_2^1 \cdot \det A_1^2 = \det A \cdot \det A_{1,2}^{1,2}.$$

(2) If A is a skew-symmetric matrix, then we have

$$\mathrm{Pf}\, A_{1,2}^{1,2} \cdot \mathrm{Pf}\, A_{3,4}^{3,4} - \mathrm{Pf}\, A_{1,3}^{1,3} \cdot \mathrm{Pf}\, A_{2,4}^{2,4} + \mathrm{Pf}\, A_{1,4}^{1,4} \cdot \mathrm{Pf}\, A_{2,3}^{2,3} = \mathrm{Pf}\, A \cdot \mathrm{Pf}\, A_{1,2,3,4}^{1,2,3,4}.$$

Theorem (A homogeneous version, a special case)

For six vectors of variables

$$X = (x_1, \cdots, x_{2n}),\ Y = (y_1, \cdots, y_{2n}),\ A = (a_1, \cdots, a_{2n}),$$
$$B = (b_1, \cdots, b_{2n}),\ C = (c_1, \cdots, c_{2n}),\ D = (d_1, \cdots, d_{2n}),$$

we have

$$\operatorname*{Pf}_{1 \le i < j \le 2n} \left[\frac{\begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} \cdot \begin{vmatrix} c_i & d_i \\ c_j & d_j \end{vmatrix}}{\begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix}} \right] = \frac{\det U^n(X, Y; A, B) \det U^n(X, Y; C, D)}{\prod_{1 \le i < j \le 2n} \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix}}.$$

Corollary

For three vectors of variables

$$X_{2n} = (x_1, \ldots, x_{2n}),\ A_{2n} = (a_1, \ldots, a_{2n}),\ B_{2n} = (b_1, \ldots, b_{2n})$$

we have

$$\operatorname*{Pf}_{1 \le i < j \le 2n} \left[\frac{a_i b_j - a_j b_i}{1 - t x_i x_j} \right] = (-1)^{\binom{n}{2}} t^{\binom{n}{2}} \frac{\det U^n(X_{2n}, 1 + tX_{2n}^2; A_{2n}, B_{2n})}{\prod_{1 \le i < j \le 2n} (1 - t x_i x_j)},$$

where $X_{2n}^2 = (x_1^2, \ldots, x_{2n}^2)$ and $1 + tX_{2n} = (1 + x_1^2, \ldots, 1 + x_{2n}^2)$.

The aim of Step3

Prove Stanley's open problem by evaluating the determinant obtained in Theorem B (Use Stembridge's criterion).

<u>Proposition</u> (Stembridge)

Let $f(x_1, x_2, \ldots)$ be a symmetric function with infinite variables. Then

$$f \in \mathbb{Q}[p_1, p_3, p_5, \ldots]$$

if and only if

$$f(t, -t, x_1, x_2, \ldots) = f(x_1, x_2, \ldots).$$

See Stanley's book "Enumerative Combinatorics II", p.p. 450, Exercise 7.7, or Stembridge's paper "Enriched P-partitions", Trans. Amer. Math. Soc. **349** (1997), 763–788.

Sketch of the proof

Put

$$w_n(X_{2n}) = \log z_n(X_{2n}) - \sum_{k\geq 1} \frac{1}{2k} a^k (b^k - c^k) p_{2k}(X_{2n})$$
$$- \sum_{k\geq 1} \frac{1}{4k} a^k b^k c^k d^k p_{2k}(X_{2n})^2.$$

Our goal is to show

$$w_{n+1}(t, -t, X_{2n}) = w_n(X_{2n}).$$

Method

Let $X = X_{2n} = (x_1, \ldots, x_{2n})$ be a $2n$-tuple of variables. Put

$$f_n(X_{2n}) = U^n(X^2, 1 + abcdX^4; X + aX^2, 1 - a(b+c)X^2 - abcX^3).$$

Then $f_n(X_{2n})$ satisfies

$$f_{n+1}(t, -t, X_{2n}) = (-1)^n 2t \times (1 - abt^2)(1 - act^2) \times \prod_{i=1}^{2n}(t^2 - x_i^2) \prod_{i=1}^{2n}(1 - abcdt^2 x_i^2) \cdot f_n(X_{2n}).$$

The end of the proof

Corollaries and conjectures

Let $S_\lambda(x;t) = \det(q_{\lambda_i - i + j}(x;t))$ denote the big Schur function corresponding to the partition λ.

Corollary

Let

$$Z(x;t) = \sum_{\lambda} \omega(\lambda) S_\lambda(x;t),$$

Here the sum runs over all partitions λ.

Then we have

$$\log Z(x;t) - \sum_{n \geq 1} \frac{1}{2n} a^n (b^n - c^n)(1 - t^{2n}) p_{2n} \\ - \sum_{n \geq 1} \frac{1}{4n} a^n b^n c^n d^n (1 - t^{2n})^2 p_{2n}^2 \in \mathbb{Q}[[p_1, p_3, p_5, \dots]].$$

Definition

Define $T_\lambda(x;q,t)$ by

$$\det\left(Q_{(\lambda_i-i+j)}(x;q,t)\right)_{1\le i,j\le \ell(\lambda)},$$

where $Q_\lambda(x;q,t)$ stands for the Macdonald polynomial corresponding to the partition λ, and $Q_{(r)}(x;q,t)$ is the one corresponding to the one row partition (r) (See Macdonald's book, IV, sec.4).

Corollary

Let

$$Z(x;q,t)=\sum_{\lambda}\omega(\lambda)T_{\lambda}(x;q,t),$$

Here the sum runs over all partitions λ.

Then we have

$$\log Z(x;q,t)-\sum_{n\geq 1}\frac{1}{2n}a^n(b^n-c^n)\frac{1-t^{2n}}{1-q^{2n}}p_{2n}$$
$$-\sum_{n\geq 1}\frac{1}{4n}a^nb^nc^nd^n\frac{(1-t^{2n})^2}{(1-q^{2n})^2}p_{2n}^2\in\mathbb{Q}[[p_1,p_3,p_5,\dots]].$$

The Hall-Littlewood polynomials

Let $X = (x_1, \ldots, x_n)$ be variables, and let λ be a partition such that $\ell(\lambda) \le n$.

The Hall-Littlewood polynomial with respect to λ is, by definition,

$$P_\lambda(x_1, \ldots, x_n; t) = \frac{1}{v_\lambda(t)} \sum_{w \in S_n} w\left(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right).$$

Here

$$v_\lambda(t) = \prod_{i \ge 0} \frac{(1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})}{(1-t)^{m_i(\lambda)}}$$

where $m_i(\lambda)$ is the number of λ_j equal to i.

The Hall-Littlewood polynomials

$$Q_\lambda(x;t) = b_\lambda(t) P_\lambda(x;t)$$

where

$$b_\lambda(t) = \prod_{i \geq 1} \varphi_{m_i(\lambda)}(t)$$

Here $m_i(\lambda)$ denotes the number of times i occurs as a part of λ, and $\varphi_r(t) = (1-t)(1-t^2)\cdots(1-t^r)$.

It is well-known that

$$Q_\lambda(x;-1) \in \mathbb{Q}[p_1, p_3, p_5, \ldots].$$

See Macdonald's book, III, 8. Schur's Q-functions.

Conjecture

Let

$$w(x;t) = \sum_{\lambda} \omega(\lambda) P_\lambda(x;t),$$

where $P_\lambda(x;t)$ denote the Hall-Littlewood function corresponding to the partition λ, and the sum runs over all partitions λ. Then

$$\log w(x;-1) + \sum_{n\geq 1 \text{ odd}} \frac{1}{2n} a^n c^n p_{2n} + \sum_{n\geq 2 \text{ even}} \frac{1}{2n} a^{\frac{n}{2}} c^{\frac{n}{2}} (a^{\frac{n}{2}} c^{\frac{n}{2}} - 2 b^{\frac{n}{2}} d^{\frac{n}{2}}) p_{2n} \in \mathbb{Q}[[p_1, p_3, p_5, \ldots]].$$

would hold.

The Macdonald polynomials

$P_\lambda(x;q,t)$: Macdonald's P-function corresponding to the partition λ

$Q_\lambda(x;q,t)$: Macdonald's Q-function corresponding to the partition λ

See Macdonald's book IV.

Relation:

$$Q_\lambda(x;q,t) = b_\lambda(q,t)P_\lambda(x;q,t),$$

where

$$b_\lambda(q,t) = \prod_{s\in\lambda} \frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}.$$

Is it known that

$$Q_\lambda(x;q,-1) \in \mathbb{Q}(q)[p_1,p_3,p_5,\ldots]?$$

Conjecture

Let

$$w(x;q,t) = \sum_{\lambda} \omega(\lambda) P_{\lambda}(x;q,t).$$

where $P_{\lambda}(x;q,t)$ denote the Macdonald polynomial corresponding to the partition λ, and the sum runs over all partitions λ. Then

$$\log w(x;q,-1) + \sum_{n\geq 1 \text{ odd}} \frac{1}{2n} a^n c^n p_{2n} + \sum_{n\geq 2 \text{ even}} \frac{1}{2n} a^{\frac{n}{2}} c^{\frac{n}{2}} (a^{\frac{n}{2}} c^{\frac{n}{2}} - 2b^{\frac{n}{2}} d^{\frac{n}{2}}) p_{2n} \in \mathbb{Q}(q)[[p_1, p_3, p_5, \dots]]$$

would hold.

Theorem (Boulet)

$$\sum_{\mu \text{ distict partitions}} \omega(\mu) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^{j-1} c^{j-1} d^{j-1})(1 + a^j b^j c^j d^{j-1})}{1 - a^j b^j c^{j-1} d^{j-1}}$$

Here the sum runs over all distinct partitions μ.

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