The refinements of TSSCPP enumerateion

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References

- Mills-Robbins-Rumsey, “Self-complementary totally symmetric plane partitions” J. Combin. Theory Ser. A, **42** (1986), 277 – 292.
- Masao Ishikawa, “On refined enumerations of totally symmetric self-complementary plane partitions”, in preparation.

Certain Numbers

1. A_n : ASM numbers
2. A_n^r, $A_n(t)$: the refined ASM numbers.
3. $A_n^{k,l}$ $A_n(t,u)$: the doubly refined ASM numbers.
4. A_n^{HTS}, $A_n^{\mathsf{HTS}}(t)$: the number of half-turn symmetric ASMs and its refinments.
5. A_n^{VS}, $A_n^{\mathsf{VS}}(t)$: the number of ASMs invariant under the vertical flip and its refinments.

A_n

Let A_n denote the number defined by

$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

This number is famous for the number of alternating sign matrices.

A_n^r

Let n be a positive number and let $1 \le r \le n$. Set A_n^r to be the number

$$A_n^r = \frac{\binom{n+r-2}{n-1}\binom{2n-r-1}{n-1}}{\binom{2n-2}{n-1}} A_{n-1} = \frac{\binom{n+r-2}{n-1}\binom{2n-1-r}{n-1}}{\binom{3n-2}{n-1}} A_n.$$

Then the number A_n^r satisfies the recurrence $A_n^1 = A_{n-1}$ and

$$\frac{A_n^{r+1}}{A_n^r} = \frac{(n-r)(n+r-1)}{k(2n-r-1)}.$$

We also define the polynomial $A_n(t) = \sum_{r=1}^n A_n^r t^{r-1}$. For instance, the first few terms are $A_1(t) = 1$, $A_2(t) = 1 + t$, $A_3(t) = 2 + 3t + 2t^2$, $A_4(t) = 7 + 14t + 14t^2 + 7t^3$.

$A_n^{k,l}$

Let n be a positive integer and let $A_n^{k,l}$, $1 \leq k, l \leq n$, denote the number which satisfies the initial condition

$$A_n^{k,1} = A_n^{1,k} = \begin{cases} 0 & \text{if } k = 1 \\ A_{n-1}^{n-k} & \text{if } 2 \leq k \leq n \end{cases}$$

and the recurrence equation

$$A_n^{k+1,l+1} - A_n^{k,l} = \frac{A_{n-1}^{k}(A_n^{l+1} - A_n^{l}) + A_{n-1}^{l}(A_n^{k+1} - A_n^{k})}{A_n^{1}}$$

for $1 \leq k, l \leq n-1$.

Example

This recurrence equation satisfied by $A_n^{k,l}$ has been introduced by Stroganov to describe the double distribution of the positions of the 1's in the top row and the bottom row of an alternating sign matrix.

$$\left(A_4^{k,l}\right)_{1\le k,l\le 4} = \begin{pmatrix} 0 & 2 & 3 & 2 \\ 2 & 4 & 5 & 3 \\ 3 & 5 & 4 & 2 \\ 2 & 3 & 2 & 0 \end{pmatrix}.$$

$A_n(t,u)$

Let $A_n(t,u)$ denote the polynomial defined by $A_n(t,u) = \sum_{k,l=1}^{n} A_n^{k,l} t^{k-1} u^{n-l}$. Let $\omega = e^{2i\pi/3}$. Francesco and Zinn-Justin showed that $A_n(t,u)$ can be expressed by the Schur function as

$$A_n(t,u) = \frac{\{\omega^2(\omega+t)(\omega+u)\}^{n-1}}{3^{n(n-1)/2}} \times s^{(2n)}_{\delta(n-1,n-1)}\left(\frac{1+\omega t}{\omega+t}, \frac{1+\omega u}{\omega+u}, 1, \dots, 1\right)$$

where $s^{(n)}_{\lambda}(x_1,\dots,x_n)$ stands for the Schur function in the n variables $x_1, \dots, x_n$, corresponding to the partition λ, and $\delta(n-1,n-1) = (n-1, n-1, n-2, n-2, \dots, 1, 1)$

A_n^{HTS}

Let A_n^{HTS} be the number defined by

$$A_{2n}^{\mathsf{HTS}} = \prod_{i=0}^{n-1} \frac{(3i)!(3i+2)!}{\{(n+i)!\}^2}$$

and

$$A_{2n+1}^{\mathsf{HTS}} = \frac{n!(3n)!}{\{(2n)!\}^2} \cdot A_{2n}^{\mathsf{HTS}}.$$

The first few terms are $1, 2, 3, 10, 25, 140, 588.$ This is the number of half-turn symmetric alternating sign matrices.

$\widetilde{A}_n^{\mathsf{HTS}}(t)$

We also define the polynomial $\widetilde{A}_n^{\mathsf{HTS}}(t)$ by

$$\frac{\widetilde{A}_{2n}^{\mathsf{HTS}}(t)}{\widetilde{A}_{2n}^{\mathsf{HTS}}} = \frac{(3n-2)(2n-1)!}{(n-1)!(3n-1)!} \sum_{r=0}^{n} \frac{\{n(n-1)-nr+r^2\}(n+r-2)!(2n-r-2)!}{r!(n-r)!} t^r$$

where $\widetilde{A}_{2n}^{\mathsf{HTS}} = \prod_{i=0}^{n-1} \frac{(3i)!(3i+2)!}{(3i+1)!(n+i)!}$. For instance, the first few terms are $\widetilde{A}_2^{\mathsf{HTS}}(t) = 1 + t$, $\widetilde{A}_4^{\mathsf{HTS}}(t) = 2 + t + 2t^2$,
$\widetilde{A}_6^{\mathsf{HTS}}(t) = 5 + 5t + 5t^2 + 5t^3$ and
$\widetilde{A}_8^{\mathsf{HTS}}(t) = 20 + 30t + 32t^2 + 30t^3 + 20t^4$.

$A_{2n}^{\mathsf{HTS}}(t)$

Let

$$A_{2n}^{\mathsf{HTS}}(t) = \widetilde{A}_{2n}^{\mathsf{HTS}}(t) A_n(t),$$

and

$$A_{2n+1}^{\mathsf{HTS}}(t) = \frac{1}{3}\left\{ A_{n+1}(t)\widetilde{A}_{2n}^{\mathsf{HTS}}(t) + A_n(t)\widetilde{A}_{2n+2}^{\mathsf{HTS}}(t) \right\}.$$

The first few terms are $A_2^{\mathsf{HTS}}(t) = 1 + t$, $A_3^{\mathsf{HTS}}(t) = 1 + t + t^2$,
$A_4^{\mathsf{HTS}}(t) = 2 + 3t + 3t^2 + 2t^3$,
$A_5^{\mathsf{HTS}}(t) = 3 + 6t + 7t^2 + 6t^3 + 3t^4$. Let $A_{n,r}^{\mathsf{HTS}}$ denote the coefficient of t^r in $A_n^{\mathsf{HTS}}(t)$.

A^{VS}_{2n+1}

Let A^{VS}_{2n+1} be the number defined by

$$A^{\mathsf{VS}}_{2n+1} = \frac{1}{2^n} \prod_{k=1}^{n} \frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!}$$

and let $A^{\mathsf{VS}}_{2n+1,r}$ be the number given by

$$A^{\mathsf{VS}}_{2n+1,r} = \frac{A^{\mathsf{VS}}_{2n-1}}{(4n-2)!} \sum_{k=1}^{r} (-1)^{r+k} \frac{(2n+k-2)!(4n-k-1)!}{(k-1)!(2n-k)!}.$$

This number A^{VS}_{2n+1} is equal to the number of vertically symmetric alternating sign matrices of size $2n+1$. For example, the first few terms of A^{VS}_{2n+1} is 1, 3, 26, 646 and 45885.

$A^{\mathsf{VS}}_{2n+1}(t)$

We also define the polynomial $A^{\mathsf{VS}}_{2n+1}(t)$ by

$$A^{\mathsf{VS}}_{2n+1}(t) = \sum_{r=1}^{2n} A^{\mathsf{VS}}_{2n+1,r} t^{r-1}.$$

For instance, the first few terms are $A^{\mathsf{VS}}_{3}(t) = 1$, $A^{\mathsf{VS}}_{5}(t) = 1 + t + t^2$, $A^{\mathsf{VS}}_{7}(t) = 3 + 6t + 8t^2 + 6t^3 + 3t^4$ and $A^{\mathsf{VS}}_{9}(t) = 26 + 78t + 138t^2 + 162t^3 + 138t^4 + 78t^5 + 26t^6$.

Monotone triangles

A monotone triangle of size n is, by definition, a triangular array of positive integers

$$\begin{array}{ccccc} & & & & m_{n,n} \\ & & & m_{n-1,n-1} & m_{n-1,n} \\ & & \iddots & \vdots & \vdots \\ m_{1,1} & & \cdots & m_{1,n-1} & m_{1,n} \end{array}$$

subject to the constraints that

(M1) $m_{ij} < m_{i,j+1}$ whenever both sides are defined,

(M2) $m_{ij} \geq m_{i+1,j}$ whenever both sides are defined,

(M3) $m_{ij} \leq m_{i+1,j+1}$ whenever both sides are defined,

(M4) the bottom row $(m_{1,1}, m_{1,2}, \ldots, m_{1,n})$ is $(1, 2, \ldots, n)$.

Let $\mathcal{M}_n$ denote the set of monotone triangles of size n.

Example

$\mathcal{M}_3$ consists of the following seven elements.

$$
\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array}
\qquad
\begin{array}{ccccc} & & 2 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array}
\qquad
\begin{array}{ccccc} & & 1 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array}
\qquad
\begin{array}{ccccc} & & 2 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array}
$$

$$
\begin{array}{ccccc} & & 3 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array}
\qquad
\begin{array}{ccccc} & & 2 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array}
\qquad
\begin{array}{ccccc} & & 3 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 3 \end{array}
$$

Matrices

Let n be a positive integer.

- Let $\hat{S}_n = (\hat{s}_{ij})_{1 \le i,j \le n}$ be the skew-symmetric matrix of size n whose (i, j)entry $\hat{s}_{ij}$ is equal to $(-1)^{j-i-1}$ for $1 \le i < j \le n$.
- Let O_n denote the $n \times n$ zero matrix.
- Let $J_n = (\delta_{i,n+1-j})_{1 \le i,j \le n}$ denote the anti-diagonal matrix where $\delta_{i,j}$ stands for the Kronecker delta function.

Examples

$$\hat{S}_6 = \begin{pmatrix} 0 & 1 & -1 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 & 1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 1 & -1 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Conjectures and Progresses

Mills-Robbins-Rumsey, "Self-complementary totally symmetric plane partitions" J. Combin. Theory Ser. A, 42 (1986), 277 – 292.

The conjectures by Mills-Robbins-Rumsey

1. Conjecture 2 : the refined TSSPP conjecture.
2. Conjecture 3 : the doubly refined TSSCPP conjecture.
3. Conjecture 4 : refined HTS TSSCPP conjecture.
4. Conjecture 6 : refined VS TSSCPP conjecture.
5. Conjecture 7, 7' : (refined) MT-TSSCPP conjecture.

Triangular shifted plane partitions

Mills, Robbins and Rumsey introduced a class $\mathcal{B}_n$ of triangular shifted plane partitions $b = (b_{ij})_{1\le i\le j}$ subject to the constraints that

(B1) the shifted shape of b is $(n-1, n-2, \dots, 1)$;

(B2) $n - i \le b_{ij} \le n$ for $1 \le i \le j \le n-1$,

and they constructed a bijection between $\mathcal{T}_n$ and $\mathcal{B}_n$. In this paper we call an element of $\mathcal{B}_n$ a triangular shifted plane partition (abbreviated as TSPP) of size n.

Example

$\mathcal{B}_1$ consists of the following 1 PPs: $\emptyset$

$\mathcal{B}_2$ consists of the following 2 PPs:

$$\boxed{2} \qquad \boxed{1}$$

$\mathcal{B}_3$ consists of the followng 7 elements:

$$\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&3\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&3\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 3&2\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 2&2\\\hline \multicolumn{1}{c|}{}&2\\\cline{2-2}\end{array}\quad
\begin{array}{|c|c|}\hline 2&2\\\hline \multicolumn{1}{c|}{}&1\\\cline{2-2}\end{array}$$

Cardinality

Theorem (Andrews)

The number of the elements of $\mathcal{B}_n$ is equal to A_n.

A statistics

In this talk, for $b = (b_{ij})_{1\le i\le j\le n-1} \in \mathcal{B}_n$, we set $b_{i,n} = n - i$ for all i and $b_{0,j} = n$ for all j by convention.

Definition (Mills, Robbins and Rumsey)

For a $b = (b_{ij})_{1\le i\le j\le n-1}$ in $\mathcal{B}_n$ and integers $r = 1, \dots, n$, let

$$U_r(b) = \sum_{t=1}^{n-r} (b_{t,t+r-1} - b_{t,t+r}) + \sum_{t=n-r+1}^{n-1} \{b_{t,n-1} > n - t\}.$$

Here $\{\dots\}$ has value 1 when the statement "$\dots$" is true and 0 otherwise. for $1 \le k \le n$,

Example $n = 7$.

7	7	7	7	7	7
	6	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

$U_1(b) = 3$, $U_2(b) = 1$, $U_3(b) = 3$, $U_4(b) = 2$, $U_5(b) = 2$, $U_6(b) = 3$, $U_7(b) = 3$.

Example

$\mathcal{B}_3$ consists of the followng 7 elements:

$$
\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 3 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array}
\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array}
$$

$U_1(b)$	2	1	0	2	1	1	0
$U_2(b)$	2	2	1	1	0	1	0
$U_3(b)$	2	2	1	1	0	1	0

Refined TSSCPP conjecture

Conjecture

Let n be a positive integer. Let $1 \le k \le n$ and $1 \le r \le n$. Then the number of elements b of $\mathcal{B}_n$ such that $U_r(b) = k - 1$ would be A_n^k. Namely,

$$\sum_{b \in \mathcal{B}_n} t^{U_r(b)} = A_n(t)$$

would hold.

Theorem

Let n and N be positive integers, and let $B_n^N(t) = (b_{ij}(t))_{0 \le i \le n-1,\ 0 \le j \le n+N-1}$ be the $n \times (n+N)$ matrix whose (i,j)th entry is

$$b_{ij}(t) = \begin{cases} \delta_{0,j} & \text{if } i = 0, \\ \binom{i-1}{j-i} + \binom{i-1}{j-i-1} t & \text{otherwise.} \end{cases}$$

Theorem

Let n be a positive integer and let N be an even integer such that $N \ge n-1$. Then

$$\sum_{b \in \mathcal{B}_n} t^{U_r(b)} = \operatorname{Pf} \begin{pmatrix} O_n & J_n B_n^N(t) \\ -{}^t B_n^N(t) J_n & \hat{S}_{n+N} \end{pmatrix}.$$

Example

For example, if $n = 3$ and $N = 2$ then the above Pfaffian looks like as follows.

$$\mathrm{Pf}\left(\begin{array}{ccc|ccccc}
0 & 0 & 0 & 0 & 0 & 1 & 1+t & t \\
0 & 0 & 0 & 0 & 1 & t & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 0 & 1 & -1 & 1 & -1 \\
0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 \\
-1 & -t & 0 & 1 & -1 & 0 & 1 & -1 \\
-1-t & 0 & 0 & -1 & 1 & -1 & 0 & 1 \\
-t & 0 & 0 & 1 & -1 & 1 & -1 & 0
\end{array}\right).$$

Doubly refined TSSCPP conjecture

Conjecture

Let $n \geq 2$ and $1 \leq k, l \leq n$ be integers. Then the number of elements b of $\mathcal{B}_n$ such that $U_1(b) = k - 1$ and $U_2(b) = n - l$ would be $A_n^{k,l}$, i.e.

$$\sum_{b \in \mathcal{B}_n} t^{U_1(b)} u^{U_2(b)} = A_n(t, u).$$

Let n and N be positive integers. Let $B_n^N(t, u) = (b_{ij}(t, u))_{0 \leq i \leq n-1,\ 0 \leq j \leq n+N-1}$ be the $n \times (n + N)$ matrix whose (i, j)th entry is

Theorem

$$b_{ij}(t,u) = \begin{cases} \delta_{0,j} & \text{if } i = 0, \\ \delta_{0,j-i} + \delta_{0,j-i-1} tu & \text{if } i = 1, \\ \binom{i-2}{j-i} + \binom{i-2}{j-i-1}(t+u) + \binom{i-2}{j-i-2} tu & \text{otherwise.} \end{cases}$$

Theorem

Let n be a positive integer and let N be an even integer such that $N \geq n-1$. If r is an integer such that $2 \leq r \leq n$, then we have

$$\sum_{b \in \mathcal{B}_n} t^{U_1(b)} u^{U_r(b)} = \operatorname{Pf} \begin{pmatrix} O_n & J_n B_n^N(t,u) \\ -{}^t B_n^N(t,u) J_n & \hat{S}_{n+N} \end{pmatrix}$$

Flips

Mills Robbins and Rumsey have defined an involution $\pi_r : \mathcal{B}_n \longrightarrow \mathcal{B}_n$.

Let

$$\rho = \pi_2 \pi_4 \cdots .$$

They conjectured that the invariants of ρ in $\mathcal{B}_n$ correspond to the half-turn symmetric alternating sign matrices.

Refined HTS TSSCPP conjecture

Conjecture

Let $n \geq 2$ and r, $0 \leq r < n$ be integers. Then

$$\sum_{\substack{b \in \mathcal{B}_n \\ \rho(b)=b}} t^{U_1(b)} = A_n^{\mathsf{HTS}}(t)$$

would hold.

Result

There is a bijection between the invariants of $\mathcal{B}_n$ under ρ and a certain class of column-strict “almost domino plane partitions”.

Refined VS TSSCPP conjecture

Let

$$\gamma = \pi_1 \pi_3 \cdots .$$

They also proposed a conjecture that the invariants of γ correspond to the vertically symmetric alternating sign matrices.

Conjecture

Let $n \geq 1$ be an integer and r, $1 \leq r \leq 2n-1$ be an integer. Then

$$\sum_{\substack{b \in \mathcal{B}_{2n+1} \\ \gamma(b)=b}} t^{U_2(b)} = A^{\mathsf{VS}}_{2n+1}(t)$$

would hold.

Theorem

Let $D_n(t) = (d_{ij}(t))_{1\le i,j\le n}$ be the $n \times n$ matrix where

$$d_{ij}(t) = \binom{i+j-1}{2j-i} + \left\{ \binom{i+j-1}{2j-i+1} + \binom{i+j-1}{2j-i-1} \right\} t + \binom{i+j-1}{2j-i} t^2$$

Theorem

Let $n \geq 2$ be a positive integer. Then

$$\sum_{\substack{b\in\mathcal{B}_{2n+1} \\ \gamma(b)=b}} t^{U_2(b)} = \det D_n(t)$$

Example

Especially, when $t = 1$, this determinant becomes $\det\left(\binom{i+j-1}{2j-i+1}\right)_{1\le i,j\le n}$, and we obtain the result that the number of elements $b \in \mathcal{B}_{2n+1}$ invariant under γ is equal to A^{VS}_{2n+1} from Andrews' result (G.E. Andrews, "Pfaff's method (I): the Mills-Robbins-Rumsey determinant", Discrete Math. **193** (1998), 43–60.).

$$D_3(t) = \begin{pmatrix} 1+t+t^2 & t & 0 \\ 1+2\,t+t^2 & 3+4\,t+3\,t^2 & 1+4\,t+t^2 \\ t & 4+7\,t+4\,t^2 & 10+15\,t+10\,t^2 \end{pmatrix}$$

MT-TSSCPP conjecture

For $k = 0, 1, \ldots, n-1$, let $\mathcal{M}_n^k$ denote the set of monotone triangles with all entries m_{ij} in the first $n-k$ columns equal to their minimum values $j-i+1$. For $k = 0, 1, \ldots, n-1$, let $\mathcal{B}_n^k$ be the subset of those b in $\mathcal{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximal values n.

Conjecture

For $n \geq 2$ and $k = 0, 1, \ldots, n-1$, the cardinality of $\mathcal{B}_n^k$ is equal to the cardinality of $\mathcal{M}_n^k$.

Let n be a positive integer and let $k = 0, 1, \ldots, n-1$. Let N be an even integer such that $N \geq k$. Let
$B_n^{(k),N} = \left(b_{ij}^k\right)_{0 \leq i \leq n-1,\ 0 \leq j \leq n+N-1}$ be the $n \times (n+K)$ rectangular matrix whose $(i,j)$the entry is

Theorem

$$b_{ij}^{k} = \begin{cases} \binom{i}{j-i} & \text{if } 0 \leq j \leq n+k-1, \\ 0 & \text{if } j \geq n+k. \end{cases}$$

Theorem

Let n be a positive integer and let $k = 0, 1, \dots, n-1$. Let N be an even integer such that $N \geq k$. The cardinality of $\mathcal{B}_n^k$ is equal to

$$\mathrm{Pf} \begin{pmatrix} O_n & J_n B_n^{(k),N} \\ -{}^t B_n^{(k),N} J_n & \hat{S}_{n+N} \end{pmatrix}$$

Plane Partitions

1. Plane parttions
2. Shifted plane partitions
3. Domino plane partitions

Plane Partitions

A plane partition is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers such that π has finite support (i.e., finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum_{i,j\geq 1} \pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n, or π has the weight n.

A part of a plane partition $\pi = (\pi_{ij})_{i,j\geq 1}$ is a positive entry $\pi_{ij} > 0$. The shape of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row. The shape of π is denoted by $\mathrm{sh}(\pi)$. We say that π has r rows if $r = \ell(\lambda)$. Similarly, π has s columns if $s = \ell(\lambda')$.

Example

The following is a plane aprtition of shape $(9, 8, 4, 1)$, 4 rows, 9 columns, weight 49.

5	5	4	3	3	2	2	2	1
4	4	2	2	1	1	1	1	
2	1	1	1					
1								

Example

Plane partition of 0: $\emptyset$

Plane partition of 1: $\boxed{1}$

Plane partition of 2:

$$\begin{array}{|c|}\hline 2\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}$$

Plane partition of 3:

$$\begin{array}{|c|}\hline 3\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 1\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 2&1\\\hline\end{array} \qquad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \qquad \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{1}\\\hline 1&\\\cline{1-1}\end{array}$$

Column-strict plane partitions

A plane partition is said to be column-strict if it is weakly decreasing in rows and strictly decreasing in coulumns.

Example

5	5	4	3	3	3	1
4	4	2	2	1	1	
3	2	1	1			
1	1					

is a column-stric plane partition.

Ferrers graph

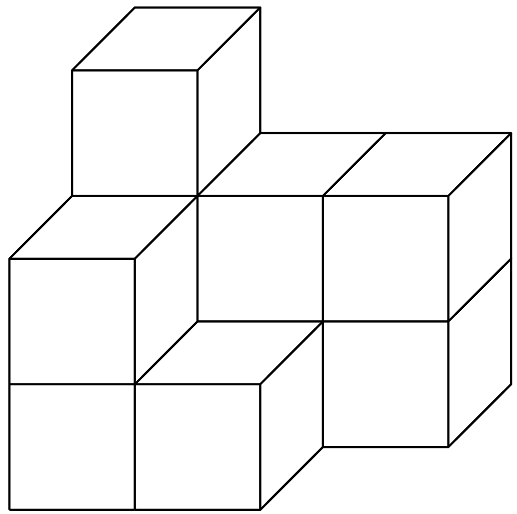
The Ferrers graph $F(\pi)$ of π is the set of all lattice points $(i, j, k) \in \mathbb{P}^3$ such that $k \leq \pi_{ij}$.

Example

The Ferrers graph of

3	2	2
2	1	

is as follows:



Shifted plane partitions

We can define a shifted plane partition similarly. A **shifted plane partition** is an array $\tau = (\tau_{ij})_{1 \le i \le j}$ of nonnegative integers such that τ has finite support and is weakly decreasing in rows and columns. The **shifted shape** of τ is the distinct partition μ for which τ has μ_i nonzero parts in the ith row.

Example

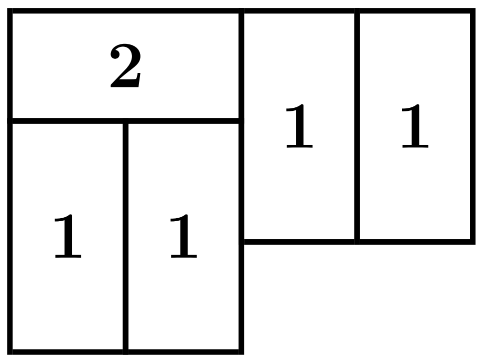
4	4	3	3	2	1	1
	4	3	2	1	1	
		2	2	1	1	
			1			

.

Domino plane partitions

Let λ be a partition. A domino plane partition of shape λ is a tiling of this shape by means of dominoes (2×1 or 1×2 rectangles), where each domino is numbered by a positive integer and those intergers are weakly decreasing in rows and columns. The integers in the dominoes are called parts. A domino plane partition is said to be column-strict if it is strictly decreasing in columns.

Example



Symmetries

1. Self-complementary plane parttions
2. Totally symmetric plane parttions

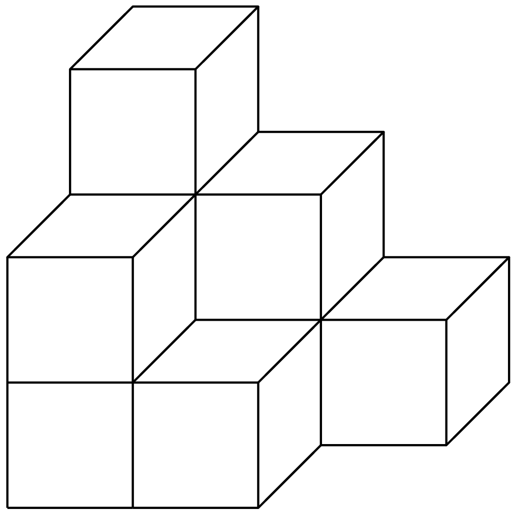
Self-complementary plane partitions

A plane partition $\pi = (\pi_{ij})_{i,j\geq 1}$ is said to be (r,c,t)-self-complementary if $\pi_{ij} = t - \pi_{r+1-i,c+1-j}$ for all $1 \leq i \leq r$ and $1 \leq j \leq c$.

Example

3	2	1
2	1	

is a $(3, 2, 3)$-self-complementary plane partition and its Ferrers graph is as follows:



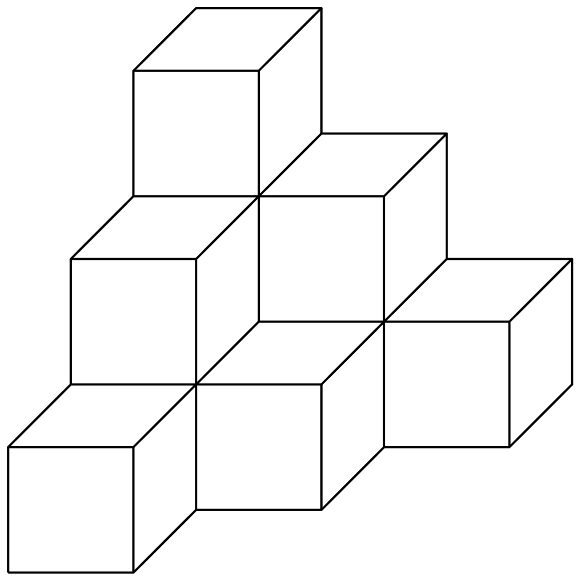
Totally symmetric plane partitions

Let P denote the set of positive integers. Consider the elements of $\mathbb{P}^3$, regarded as the lattice points of $\mathbb{R}^3$ in the positive orthant. The symmetric group S_3 is acting on $\mathbb{P}^3$ as permutations of the coordinate axies. A plane partition is said to be totally symmetric if its Ferrors graph is mapped to itself under all 6 permutations in S_3.

Example

3	2	1
2	1	
1		

is a totally symmetric plane partition and its Ferrers graph is as follows:



Our Methods

1. Generalizations and bijections
2. Description of the statistics
3. The flips and a deformed Bender-Knuth involution
4. Generating functions
   (a) Lattice paths and minor summation formulas
   (b) Invariants and domino plane partitions

Generalizations and bijections

1. Totally symmetric self-complementary plane parttions
2. Triangular shifted plane partitions
3. Restricted column-stricted plane partitions

Restricted Column-Strict Plane Partitions

Definition

Let m and $n \geq 1$ be nonnegative integers. Let $\mathcal{P}_{n,m}$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \leq i,j}$ subject to the constraints that

(C1) c has at most n columns;

(C2) c is column-strict and each part in the jth column does not exceed $n + m - j$.

We call an element of $\mathcal{P}_{n,m}$ a restricted column-stricted plane partition (abbreviated to RCSPP). When $m = 0$, we write $\mathcal{P}_n$ for $\mathcal{P}_{n,0}$.

$\mathcal{P}_{n,m}$

Example

$\mathcal{P}_1$ consists of the following 1 PPs: $\emptyset$

$\mathcal{P}_2$ consists of the following 2 PPs:

$$\emptyset \qquad \begin{array}{|c|}\hline \color{red}{1} \\ \hline \end{array}$$

$\mathcal{P}_3$ consists of the following 7 PPs:

$$\emptyset \qquad \begin{array}{|c|}\hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & \color{red}{1} \\ \hline \end{array} \qquad \begin{array}{|c|}\hline \color{red}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline \color{red}{2} & \color{red}{1} \\ \hline \end{array} \qquad \begin{array}{|c|}\hline \color{red}{2} \\ \hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline \color{red}{2} & \color{red}{1} \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

$\mathcal{P}^{\mathsf{R}}_{n,m}$ and $\mathcal{P}^{\mathsf{C}}_{n,m}$

Definition

Let $\mathcal{P}^{\mathsf{R}}_{n,m}$ denote the set of plane partitions c in $\mathcal{P}_{n,m}$ where each row has even length. and let $\mathcal{P}^{\mathsf{C}}_{n,m}$ denote the set of plane partitions c in $\mathcal{P}_{n,m}$ with each column of even length. We also write $\mathcal{P}^{\mathsf{R}}_{n}$ (resp. $\mathcal{P}^{\mathsf{C}}_{n}$) for $\mathcal{P}^{\mathsf{R}}_{n,0}$ (resp. $\mathcal{P}^{\mathsf{C}}_{n,0}$).

Example

$\mathcal{P}^{\mathsf{R}}_{3}$ consists of the following 3 PPs:

$$\emptyset \qquad \boxed{1}\boxed{1} \qquad \boxed{2}\boxed{1}$$

$\mathcal{P}_3^{\mathsf{C}}$ consists of the following 2 PPs:

$\emptyset$ $\quad\quad$ $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$

Triangular shifted plane partitions

Definition

Let m and $n \geq 1$ be nonnegative integers. Let $\mathcal{B}_{n,m}$ denote the set of shifted plane partitions $b = (b_{ij})_{1 \leq i \leq j}$ subject to the constraints that

(B1) the shifted shape of b is
$$(n + m - 1, n + m - 2, \ldots, 2, 1);$$

(B2) $\max\{n - i, 0\} \leq b_{ij} \leq n$ for $1 \leq i \leq j \leq n + m - 1$.

When $m = 0$, we write $\mathcal{B}_n$ for $\mathcal{B}_{n,0}$. In this paper we call an element of $\mathcal{B}_{n,m}$ a triangular shifted plane partition (abbreviated to TSPP).

$\mathcal{B}_{1,2}$

Example

When $n = 1$ and $m = 2$, $\mathcal{B}_{1,2}$ consists of the followng 4 elements:

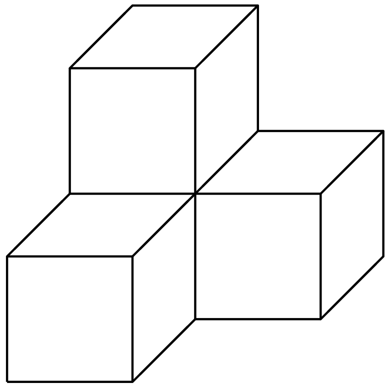
$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 0 \\ \cline{2-2} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 0 \\ \hline \multicolumn{1}{c|}{} & 0 \\ \cline{2-2} \end{array} \qquad \begin{array}{|c|c|}\hline 0 & 0 \\ \hline \multicolumn{1}{c|}{} & 0 \\ \cline{2-2} \end{array}$$

Totally symmetric self-complementary plane partitions

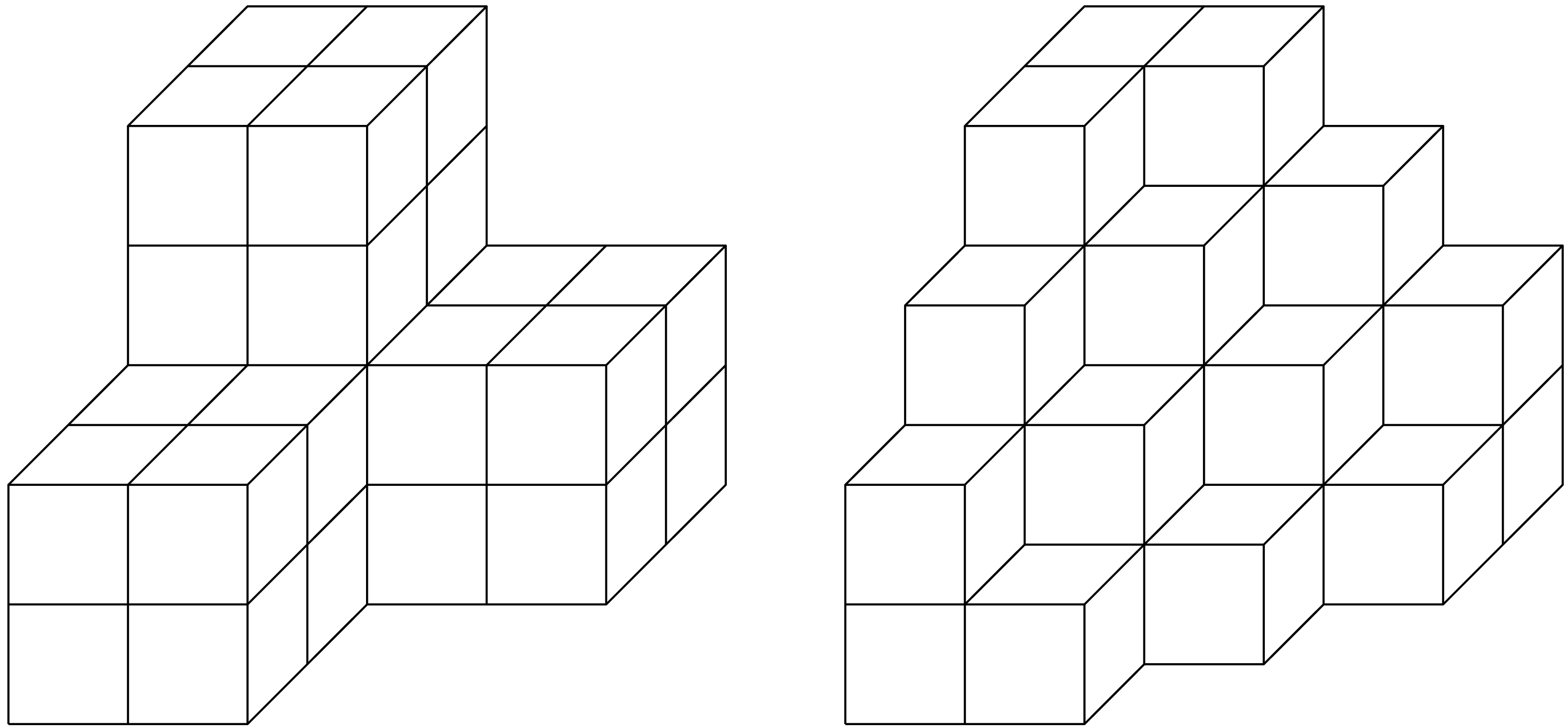
Let $\mathcal{T}_n$ denote the set of all plane partitions which is contained in the box $X_n = [2n] \times [2n] \times [2n]$, $(2n, 2n, 2n)$-self-complementary and totally symmetric. An element of $\mathcal{T}_n$ is called a totally symmetric self-complementary plane partition (abbreviated as TSSCPP) of size n.

Example

$\mathcal{T}_1$



$\mathcal{T}_2$



$\mathcal{T}_3$

6	6	6	3	3	3
6	6	6	3	3	3
6	6	6	3	3	3
3	3	3			
3	3	3			
3	3	3			

6	6	6	4	3	3
6	6	6	3	3	3
6	6	5	3	3	2
4	3	3	1		
3	3	3			
3	3	2			

6	6	6	4	3	3
6	6	6	4	3	3
6	6	4	3	2	2
4	4	3	2		
3	3	2			
3	3	2			

6	6	6	5	4	3
6	6	5	3	3	2
6	5	5	3	3	1
5	3	3	1	1	
4	3	3	1		
3	2	1			

6	6	6	5	4	3
6	6	5	4	3	2
6	5	4	3	2	1
5	4	3	2	1	
4	3	2	1		
3	2	1			

6	6	6	5	5	3
6	5	5	3	3	1
6	5	5	3	3	1
5	3	3	1	1	
5	3	3	1	1	
3	1	1			

6	6	6	5	5	3
6	5	5	4	3	1
6	5	4	3	2	1
5	4	3	2	1	
5	3	2	1	1	
3	1	1			

Terminology

Let $X_n = [n]^3$ denote the $n \times n \times n$ box. Assume n is even. We divide this box into the eight regions X_n^{+++}, X_n^{++-}, X_n^{+-+}, X_n^{+--}, X_n^{-++}, X_n^{-+-}, X_n^{--+} and X_n^{---} depending on each of $x - n/2$, $y - n/2$ and $z - n/2$ is plus (> 0) or minus (≤ 0). For example $X_n^{-+-} = [1, n/2] \times [n/2+1, n] \times [1, n/2]$. Further we use the notation $X_n^{+} = X_n^{+++} \uplus X_n^{++-} \uplus X_n^{+-+} \uplus X_n^{-++}$ and $X_n^{-} = X_n^{+--} \uplus X_n^{-+-} \uplus X_n^{--+} \uplus X_n^{---}$. More generally we write $X_n(a) = [a - n/2 + 1, a + n/2] \times [a - n/2 + 1, a + n/2] \times [a - n/2 + 1, a + n/2]$ for the $n \times n \times n$ box centered at (a, a, a).

A subclass of TSSCPPs

We also use the notation $X_n^{\pm\pm\pm}(a)$ as the same meaning as above where each stands for one of the eight regions of $X_n(a)$. For example $X_n^{+-+}(a) = [a+1, a+n/2] \times [a-n/2+1, a] \times [a+1, a+n/2]$. The symbols $X_n^{\pm}(a)$ should be defined similarly.

Definition

For nonnegative integers m and $n \geq 1$, let $\mathcal{T}_{n,m}$ denote the set of TSSCPPs $\pi \in \mathcal{T}_{n+m}$ of size $(n+m)$ which satisfy

(T) each $p \in \pi \cap X_{2m}(n)$ must be contained in $X_{2(n+m)}^{-}$.

Bijections

Theorem

There are bijections between these 3 sets of plane partitions.

$$\mathcal{P}_{n,m} \leftrightarrow \mathcal{T}_{n,m} \leftrightarrow \mathcal{B}_{n,m}$$

Cardinality

<u>Theorem</u> (Krattenthaler)

Let $m \geq 0$ and $n \geq 1$ be non-negative integers.

$$\sharp\mathcal{P}_{n,m} = \prod_{k=0}^{n-1} \frac{(3k+3m+1)!\prod_{i=0}^{m}(k+2i)!}{(2k+m)!(2k+3m+1)!\prod_{i=1}^{m}(k+2i-1)!}$$

(C. Krattenthaler, “Determinant identities and a generalization of the number of totally symmetric self-complementary plane partitions”, Electron. J. Combin. **4(1)** (1997), #R27.)

<u>Theorem</u>

Let $m \geq 0$ and $n \geq 1$ be non-negative integers.

$$\sharp\mathcal{P}_{n,m} = \sharp\mathcal{P}^{\mathsf{C}}_{n,m+1}.$$

Conjecture

Let $n \geq 1$, $r = 0, 1$ and m be nonnegative integers. Let $f(n, m)$ denote

$$\frac{\left(6n + 6\left\lfloor \frac{m}{2} \right\rfloor + 4\right)!\,\left(6n + 6\left\lceil \frac{m}{2} \right\rceil + 4\right)!(2n+1)!\,\left(2n + 2\left\lceil \frac{m}{2} \right\rceil\right)!}{(4n+m+1)!(4n+m+3)!(4n+3m+2)!(4n+3m+4)!}$$
$$\times \frac{(2n+2m+1)!\,\left(n + \left\lfloor \frac{m}{2} + 1 \right\rfloor\right)!}{\left(n + \left\lfloor \frac{m}{2} \right\rfloor\right)!\,\left(2n + 2\left\lceil \frac{m}{2} \right\rceil + 1\right)!}.$$

Then the number of elements c in $\mathcal{P}^{\mathsf{R}}_{2n+r,m}$ would be

$$2^{-n}\frac{g(n, m+r)}{g(0, m+r)}\prod_{k=0}^{n-1} f(k, m+r)$$

where

$$g(n,m) = \begin{cases} h_m(n) & \text{if } \mathrm{rem}(m,4) = 0 \text{ or } 1, \\ (4n+2m+1)h_m(n) & \text{if } \mathrm{rem}(m,4) = 2 \text{ or } 3, \end{cases}$$

and $h_m(n)$ is a polynomial of degree $2\left\lfloor \frac{m}{4} \right\rfloor$ in the variable n.

The polynomial $h_m(n)$

For small m, $h_0(n) = h_1(n) = h_2(n) = h_3(n) = 1$, $h_4(n) = 26n^2 + 117n + 132$, $h_5(n) = 94n^2 + 517n + 715$, $h_6(n) = 526n^2 + 3419n + 5610$, $h_7(n) = 2062n^2 + 15465n + 29393$, $h_8(n) = 18788n^4 + 319396n^3 + 2042275n^2 + 5821157n + 6240360$, $h_9(n) = 8564n^4 + 162716n^3 + 1163679n^2 + 3712391n + 4457400$, and so on.

Example

For example, if $m = 6$ and $r = 1$, then the number of c in $\mathcal{P}^{\mathsf{R}}_{2n+1,6}$ would be equal to

$$2^{-n}\prod_{k=0}^{n-1}\frac{(6k+22)!(6k+28)!(2k+1)!(2k+8)!(2k+15)!}{(4k+8)!(4k+10)!(4k+23)!(4k+25)!(2k+9)!}$$
$$\times\frac{(k+4)!}{(k+3)!}$$
$$\times\frac{(4n+15)(2062n^2+15465n+29393)}{15\cdot 29393}$$

and the first few terms are $\sharp\mathcal{P}^{\mathsf{R}}_{3,6} = 3432$, $\sharp\mathcal{P}^{\mathsf{R}}_{5,6} = 65934024$ and $\sharp\mathcal{P}^{\mathsf{R}}_{7,6} = 9034911255456$.

Description of the statistics

1. The statistics for $\mathcal{B}_{n,m}$
2. Saturated parts
3. The corresponding statistics for $\mathcal{P}_{n,m}$

The statistics U_r for $\mathcal{B}_{n,m}$

Definition

For a $b = (b_{ij})_{1 \le i \le j \le n+m-1}$ in $\mathcal{B}_{n,m}$ and integers $r = 1, \ldots, n$, let

$$U_r(b) = \sum_{t=1}^{n+m-r} (b_{t,t+r-1} - b_{t,t+r}) + \sum_{t=n+m-r+1}^{n+m-1} \{b_{t,n+m-1} > n-t\}.$$

This $U_r(b)$ agrees with the former definition. when $m = 0$. It is easy to check that each of these functions U_r can vary between 0 and $n + m - 1$ as b varies over $\mathcal{B}_{n,m}$. We put
$\overline{U}_r(b) = n + m - 1 - U_r(b)$.

Saturated parts

Let $\pi \in \mathcal{P}_{n,m}$. A part π_{ij} of π is said to be saturated if $\pi_{ij} = n + m - j$. A saturated part, if it exists, appears only in the first row.

Example

$$n = 7,\ m = 0.$$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

A statistics

Definition

For $\pi \in \mathcal{P}_{n,m}$ let

$$\overline{U}_k(\pi) = \sharp\{(i,j)|\pi_{ij} = k\} + \sharp\{1 \le i < k|\pi_{1,n-i} = i\}$$

for $1 \le k \le n$, i.e. $\overline{U}_k(\pi)$ is the number of parts equal to k plus the number of saturated parts less than k.

Especially,

$\overline{U}_1(\pi)$: the number of 1s in π,

$\overline{U}_n(\pi)$: the number of saturated parts in π.

Example $n = 7$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

$\overline{U}_1(\pi) = 3,\quad \overline{U}_2(\pi) = 5,\quad \overline{U}_3(\pi) = 3,\quad \overline{U}_4(\pi) = 4,$
$\overline{U}_5(\pi) = 4,\quad \overline{U}_6(\pi) = 3,\quad \overline{U}_7(\pi) = 3.$

Two new statistics $V^{\mathsf{R}}(c)$ and $V^{\mathsf{C}}(c)$

Definition

Let $c \in \mathcal{P}_{n,m}$.

(i) Let $V^{\mathsf{R}}(c)$ denote the number of rows of c of odd length.

(ii) Let $V^{\mathsf{C}}(c)$ denote the number of columns of c of odd length.

Example

For example, $\mathcal{P}_3$ consists of the following 7 elements:

$$\emptyset\;, \quad 1\;, \quad 1\;\mathbf{1}\;, \quad \mathbf{2}\;, \quad \mathbf{2}\;\mathbf{1}\;, \quad \begin{array}{l}\mathbf{2}\\ 1\end{array}\;, \quad \begin{array}{ll}\mathbf{2} & \mathbf{1}\\ 1 & \end{array}\;.$$

$\overline{U}_1(c)$	0	1	2	0	1	1	2
$\overline{U}_2(c)$	0	0	1	2	1	1	2
$\overline{U}_3(c)$	0	0	1	2	1	1	2
$V^{\mathsf{R}}(c)$	0	1	0	1	0	2	1
$V^{\mathsf{C}}(c)$	0	1	2	1	2	0	1

Table 1: The distribution statistics table in $\mathcal{P}_3$

Theorem

Let $m \geq 0$ and $n \geq 1$ be non-negative integers.

(i) If $1 \leq r, s \leq n$, then we have

$$\sum_{c \in \mathcal{P}_{n,m}} t^{\overline{U}_r(c)} = \sum_{c \in \mathcal{P}^{\mathsf{C}}_{n,m+1}} t^{\overline{U}_s(c)}.$$

(ii) If $m \geq 1$ and $2 \leq r, s \leq n$, then we have

$$\sum_{c \in \mathcal{P}_{n,m}} t^{\overline{U}_1(c)} u^{\overline{U}_r(c)} = \sum_{c \in \mathcal{P}^{\mathsf{C}}_{n,m+1}} t^{\overline{U}_1(c)} u^{\overline{U}_s(c)}.$$

A new conjecture

Conjecture

Let $n \geq 1$ be a positive integer, and let $1 \leq r \leq n$. Then

$$\sum_{c \in \mathcal{P}_n^{\mathsf{R}}} t^{\overline{U}_r(c)} = \begin{cases} A_{2m+1}^{\mathsf{VS}} \cdot A_{2m+1}^{\mathsf{VS}}(t) & \text{if } n = 2m, \\ A_{2m+1}^{\mathsf{VS}} \cdot A_{2m+3}^{\mathsf{VS}}(t) & \text{if } n = 2m+1. \end{cases}$$

would hold. Especially, if we put $t = 1$, the number of c in $\mathcal{P}_n^{\mathsf{R}}$ is even would be

$$\begin{cases} \left(A_{2m+1}^{\mathsf{VS}}\right)^2 & \text{if } n = 2m, \\ A_{2m+1}^{\mathsf{VS}} \cdot A_{2m+3}^{\mathsf{VS}} & \text{if } n = 2m+1. \end{cases}$$

The flips and a deformed Bender-Knuth involution

1. Flips on $\mathcal{B}_{n,m}$
2. Deformed Bender-Knuth involution on $\mathcal{P}_{n,m}$
3. Involutions corresponding the half-turn and the vertical flip

Flip

Let $b = (b_{ij})_{1 \le i \le j \le n+m-1}$ be an element of $\mathcal{B}_{n,m}$ and let $1 \le i < j \le n+m-1$ so that b_{ij} is a part of b off the main diagonal. Then the flip of the part b_{ij} is the operation of replacing b_{ij} by b'_{ij} where

$$b'_{ij} + b_{ij} = \min(b_{i-1,j}, b_{i,j-1}) + \max(b_{i,j+1}, b_{i+1,j}).$$

When the part is in the main diagonal, the flip of a part b_{ii} is the operation replacing b_{ii} by b'_{ii} where

$$b'_{ii} + b_{ii} = b_{i-1,i} + b_{i,i+1}.$$

Involution

Let $1 \leq r \leq n+m$ and $b = (b_{ij})_{1\leq i\leq j\leq n+m-1} \in \mathcal{B}_{n,m}$. Define an opration

$$\pi_r : \mathcal{B}_{n,m} \longrightarrow \mathcal{B}_{n,m}$$
$$b \longmapsto \pi_r(b)$$

where $\pi_r(b)$ is the result of flipping all the $b_{i,i+r-1}$, $1 \leq i \leq n+m-r$. Since none of these parts of b are neighbors, the result is indpendent of the order in which the flips are applied, and this operation π_r is evidently an involution, i.e. $\pi_r^2 = id$.

Example

The seven elements of $\mathcal{B}_3$

$$\begin{matrix} 3 & 3 \\ & 3 \end{matrix} \quad \begin{matrix} 3 & 3 \\ & 2 \end{matrix} \quad \begin{matrix} 3 & 3 \\ & 1 \end{matrix} \quad \begin{matrix} 3 & 2 \\ & 2 \end{matrix} \quad \begin{matrix} 3 & 2 \\ & 1 \end{matrix} \quad \begin{matrix} 2 & 2 \\ & 2 \end{matrix} \quad \begin{matrix} 2 & 2 \\ & 1 \end{matrix}$$

is mapped to

$$\begin{matrix} 3 & 3 \\ & 1 \end{matrix} \quad \begin{matrix} 3 & 3 \\ & 2 \end{matrix} \quad \begin{matrix} 3 & 3 \\ & 3 \end{matrix} \quad \begin{matrix} 2 & 2 \\ & 1 \end{matrix} \quad \begin{matrix} 2 & 2 \\ & 2 \end{matrix} \quad \begin{matrix} 3 & 2 \\ & 1 \end{matrix} \quad \begin{matrix} 3 & 2 \\ & 2 \end{matrix}$$

by π_1, respectively.

A deformed Bender-Knuth involution

Now we define a Bender-Knuth type involution $\widetilde{\pi}_r : \mathcal{P}_{n,m} \longrightarrow \mathcal{P}_{n,m}$.
Let $1 \leq r \leq n+m$ and $c \in \mathcal{P}_n$.

$\widetilde{\pi}_r$ swaps r and $r-1$ in c as the ordinary Bender-Knuth involution with one exception:

(*) We don't count the $r-1$ which is saturated.

Involution $\widetilde{\pi}_r$

Define an operation $\widetilde{\pi}_r : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ by $c \mapsto \widetilde{\pi}_r(c)$ where $\widetilde{\pi}_r(c)$ is the result of swapping r's and $r-1$'s in row i of c by this deformed rule for $1 \leq i \leq n-r$. We call the involution $\widetilde{\pi}_r$, $1 \leq r \leq n$, the deformed Bender-Knuth involution (abbreviated to the DBK involution).

$$
\begin{array}{c||ccc|cccccc|c}
i-1 & \vdots & & \vdots & & & & & & & r \\
\cline{2-4}
i & r & \ldots & r & r & \ldots & r & r-1 & \ldots & r-1 & r-1 \\
i+1 & r-1 & \ldots & r-1 & & & & & & & \\
\cline{2-4}
\end{array}
$$

Example

$n = 6$, $m = 0$ and $r = 2$

$$c = \begin{array}{ccccc} 5 & 3 & 1 & 1 & \color{red}{1} \\ 3 & 2 & & & \\ 2 & 1 & & & \end{array} \qquad \widetilde{\pi}_2(c) = \begin{array}{ccccc} 5 & 3 & 2 & 2 & \color{red}{1} \\ 3 & 2 & & & \\ 1 & 1 & & & \end{array}$$

$$\widetilde{\pi}_1(c) = \begin{array}{cc} 5 & 3 \\ 3 & 2 \\ 2 & \\ 1 & \end{array}.$$

Proposition

Let $n \geq 1$ be non-negative integers. Let $2 \leq r \leq n$ and let c in $\mathcal{P}_n$. Then

$$\overline{U}_r\left(\widetilde{\pi}_r(c)\right) = \overline{U}_{r-1}\left(c\right)$$

and

$$\overline{U}_r\left(c\right) = \overline{U}_{r-1}\left(\widetilde{\pi}_r(c)\right).$$

An involution corresponding to the half-turn

Define an involution $\widetilde{\gamma} : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ by

$$\widetilde{\rho} = \widetilde{\pi}_2 \widetilde{\pi}_4 \widetilde{\pi}_6 \cdots$$

where the product is over all $\widetilde{\pi}_i$ with i even and $\leq n$.

Let $\mathcal{P}_n^{\widetilde{\rho}}$ denote the set of elements of $\mathcal{P}_n$ which is invariant under $\widetilde{\rho}$.

Example

There are 1 elements of $\mathcal{P}_1$ that is invariant under $\widetilde{\rho}$.

$$\emptyset$$

There are 2 elements of $\mathcal{P}_2$ that is invariant under $\widetilde{\rho}$.

$$\emptyset \qquad \boxed{1}$$

There are 3 elements of $\mathcal{P}_3$ that is invariant under $\widetilde{\rho}$.

$$\emptyset \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & \color{red}{1} \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

There are 10 elements of $\mathcal{P}_4$ that is invariant under $\widetilde{\rho}$.

There are 25 elements of $\mathcal{P}_5$ that is invariant under $\widetilde{\rho}$.

An involution corresponding to the vertical flip

Define an involution $\widetilde{\gamma} : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ by

$$\widetilde{\gamma} = \widetilde{\pi}_1 \widetilde{\pi}_3 \widetilde{\pi}_5 \cdots$$

where the product is over all $\widetilde{\pi}_i$ with i odd and $\leq n$.

Let $\mathcal{P}_n^{\widetilde{\gamma}}$ denote the set of elements of $\mathcal{P}_n$ which is invariant under $\widetilde{\gamma}$.

$\mathcal{P}_n^{\widetilde{\gamma}}$ is empty unless n is odd.

Example

There are 1 element of $\mathcal{P}_3$ which is invariant under $\widetilde{\gamma}$.

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

There are 3 element of $\mathcal{P}_5$ which is invariant under $\widetilde{\gamma}$.

$$\emptyset \qquad \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 1 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & 3 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$$

There are 26 element of $\mathcal{P}_7$

Startegy

1. There is a bijection

$$\mathcal{P}_n^{\widetilde{\rho}} \leftrightarrow \{\text{certain almost domino PPs}\}$$

2. There is a bijection

$$\mathcal{P}_n^{\widetilde{\gamma}} \leftrightarrow \{\text{certain domino PPs}\}$$

Restricted column-stricted domino plane partitions

Let $\mathcal{P}^{\mathsf{VS}}_{2n+1}$ be the set of domino plane partitions c which satisfies

(F1) the shape of c is even;

(F2) c is column-strict;

(F3) each part in the jth column does not exceed $\lfloor (2n+2-j)/2 \rfloor$.

We call an element of $\mathcal{P}^{\mathsf{VS}}_{2n+1}$ a restricted column-strict domino plane partition (abbreviated to RCSDPP). The condition (F3) can be restated as follows; if $c \in \mathcal{P}^{\mathsf{VS}}_{2n+1}$, then all the parts in the 1st and 2nd row of c are $\leq n-1$, all the parts in the 3rd and 4th row of c are $\leq n-2$, and so on.

Example

For example, if n=5, then $\mathcal{P}_5^{\mathsf{VS}}$ is composed of the following three elements.

$\emptyset$

We also let $\overline{U}_1(c)$ denote the number of 1's in c for $c \in \mathcal{P}_{2n+1}^{\mathsf{VS}}$. From the above example, we have $\sum_{c\in\mathcal{P}_5^{\mathsf{VS}}} t^{\overline{U}_1(c)} = 1 + t + t^2$. The reader can easily check that there are 26 elements in $\mathcal{P}_7^{\mathsf{VS}}$ and $\sum_{c\in\mathcal{P}_7^{\mathsf{VS}}} t^{\overline{U}_1(c)} = 3 + 6t + 8t^2 + 6t^3 + 3t^4$.

A bijection

Theorem

There is a bijection between RCSPPs $\mathcal{P}_{2n+1}$ invariant under $\widetilde{\gamma}$ and RCSDPPs $\mathcal{P}^{\mathsf{VS}}_{2n+1}$. By this bijection $\overline{U}_2$ of $\mathcal{P}_{2n+1}$ corresponds to $\overline{U}_1$ of $\mathcal{P}^{\mathsf{VS}}_{2n+1}$.

Generating functions

1. The generating functions for $\mathcal{P}_{n,m}$
2. Stanton-White bijection between restricted column-strict domino plane partitions and pairs of $\mathcal{P}_{n,m}$

The lattice paths for $\mathcal{P}_{n,m}$

$c \in \mathcal{P}_{n,m}$ can be interpreted by lattice paths.

Let $t = (t_1, \ldots, t_n)$ and $x = (x_1, \ldots, x_{n-1})$ be sets of variables.
Let $\overline{U}(\pi) = (\overline{U}_1(\pi), \ldots, \overline{U}_n(\pi))$ and we set
$t^{\overline{U}(\pi)} = \prod_{k=1}^{n} t_k^{\overline{U}_k(\pi)}$. Similarly we write x^{π} for $\prod_{ij} x_{\pi_{ij}}$.

The generating functions for $\mathcal{P}_{n,m}$

Theorem

$$\sum_{\substack{\pi \in \mathcal{P}_n \\ \mathrm{sh}(\pi)=\lambda'}} t^{\overline{U}(\pi)} x^{\pi}$$

$$= \det\left(e^{(n-i)}_{\lambda_j - j + i}\left(t_1 x_1, \ldots, t_{n-i-1} x_{n-i-1}, \prod_{r=1}^{n} t_r x_{n-i} \right)\right)_{1 \le i,j \le n}$$

where $e_r^{(m)}(x)$ denote the rth elementary symmetric function in the viariables $(x_1, \ldots, x_m)$, i.e.

$$\sum_r e_r^{(m)}(x) z^r = \prod_{i=1}^{m} (1 + x_i z)$$

Corollary

$$\sum_{\pi \in \mathcal{P}_n} t^{\overline{U}(\pi)} x^{\pi}$$

is the sum of the all minors of the rectangular matrix

$$\left[e_{j-i}^{(i)} \left(t_1 x_1, \ldots, t_{n-i-1} x_{n-i-1}, \prod_{r=1}^{n} t_r x_{n-i} \right) \right]_{\substack{0 \le i \le n-1 \\ 0 \le j \le 2n-2}}$$

of size n.

Example.

When $n = 3$, the sum of all minors of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & t_1 t_2 t_3 x_1 & 0 & 0 \\ 0 & 0 & 1 & t_1 x_1 + t_2 t_3 x_2 & t_1 t_2 t_3 x_1 x_2 \end{bmatrix}$$

is $1 + t_1 x_1 + t_2 t_3 x_2 + t_1 t_2 t_3 x_1 x_2 + t_1^2 t_2 t_3 x_1^2 + t_1 t_2^2 t_3^2 x_1 x_2 + t_1^2 t_2^2 t_3^2 x_1^2 x_2$.

Each term corresponds to the following PPs:

$$\emptyset \qquad \overline{U}_1(\pi) = 0 \qquad \overline{U}_2(\pi) = 0 \qquad \overline{U}_3(\pi) = 0 \qquad 1$$

$$\begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad \overline{U}_1(\pi) = 1 \qquad \overline{U}_2(\pi) = 0 \qquad \overline{U}_3(\pi) = 0 \qquad t_1 x_1$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \overline{U}_1(\pi) = 2 \qquad \overline{U}_2(\pi) = 1 \qquad \overline{U}_3(\pi) = 1 \qquad t_1^2 t_2 t_3 x_1^2$$

$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \qquad \overline{U}_1(\pi) = 0 \qquad \overline{U}_2(\pi) = 1 \qquad \overline{U}_3(\pi) = 1 \qquad t_2 t_3 x_2$$

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \overline{U}_1(\pi) = 1 \qquad \overline{U}_2(\pi) = 2 \qquad \overline{U}_3(\pi) = 2 \qquad t_1 t_2^2 t_3^2 x_1 x_2$$

$$\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \overline{U}_1(\pi) = 1 \qquad \overline{U}_2(\pi) = 1 \qquad \overline{U}_3(\pi) = 1 \qquad t_1 t_2 t_3 x_1 x_2$$

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array} \qquad \overline{U}_1(\pi) = 2 \qquad \overline{U}_2(\pi) = 2 \qquad \overline{U}_3(\pi) = 2 \qquad t_1^2 t_2^2 t_3^2 x_1^2 x_2$$

Stanton-White bijection

Stanton-White defined a bijection between a domino plane partition T and a pair of plane partitions (T^0, T^1).

D. Stanton and D. White, "A Schensted algorithm for rim hook tableaux", J. Combin. Theory Ser. A **40** (1985), 211 – 247.

Proposition

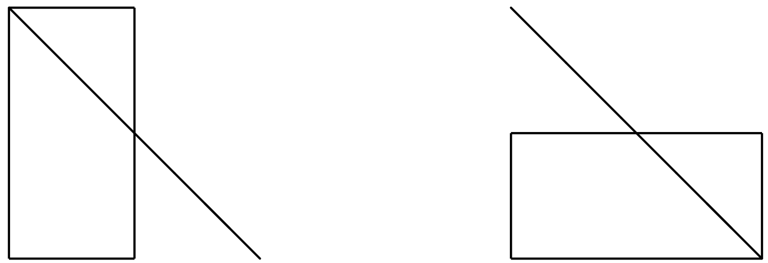
By this bijection,

1. the shape of T is even if and only if the shape T^0 is obtained by removing a vertical strip from the shape of T^1;
2. the conjugate of the shape of T is even if and only if the shape T^1 is obtained by removing a horizontal strip from the shape of T^0,

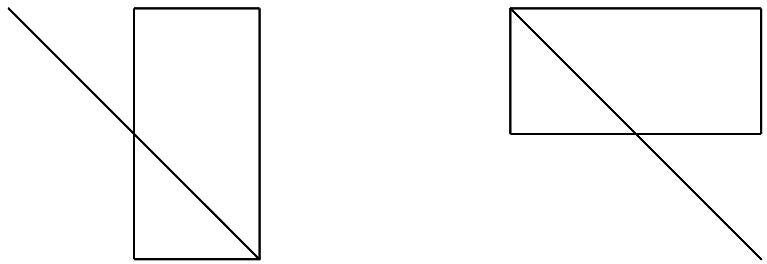
   (See also C. Carré and B. Leclerc, "Splitting the Square of a Schur Function into its Symmetric and Antisymmetric Parts", J. Algebraic Combin. **4** (1995), 201 – 231.)

Color rule

Color 0:

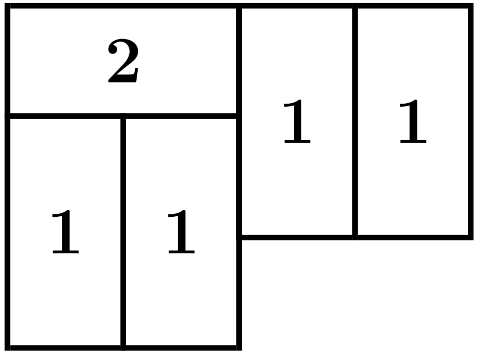


Color 1:

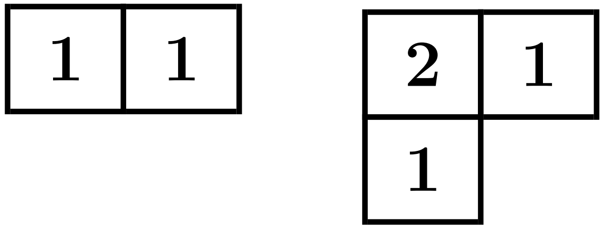


Example

The domino plane partition



correspond to the following pair of plane partitions:



Paired restriced column-stricted plane partitions

Let $\mathcal{Q}_n^{\mathsf{VS}}$ be the set of pairs (c^0, c^1) of plane partitions which satisfies

(G1) $c^0, c^2 \in \mathcal{P}_n$;

(G2) The shape of c^0 is obtained by removing a vertical strip from the shape of c^1.

We call an element of $\mathcal{Q}_n^{\mathsf{VS}}$ a paired restricted column-strict plane partition (abbreviated to PRCSPP).

Theorem

There is a bijection between RCSPPs $\mathcal{P}_n$ invariant under $\widetilde{\gamma}$ and PRCSPPs $\mathcal{Q}_n^{\mathsf{VS}}$.

Example

$\mathcal{P}_{\mathbf{5}}^{\mathsf{VS}}$ is composed of the following three elements

$$\emptyset\,, \qquad \begin{array}{|c|}\hline \mathbf{1} \\ \hline \end{array}\,, \qquad \begin{array}{|c|c|}\hline \mathbf{1} & \mathbf{1} \\ \hline \end{array}\,,$$

which corresponds to

$$(\emptyset, \emptyset), \qquad \left(\emptyset, \begin{array}{|c|}\hline \mathbf{1} \\ \hline \end{array}\right), \qquad \left(\begin{array}{|c|}\hline \mathbf{1} \\ \hline \end{array}, \begin{array}{|c|}\hline \mathbf{1} \\ \hline \end{array}\right),$$

respectively.

The generating function for $\mathcal{P}_n^{\widetilde{\gamma}}$

Let $D_n(t) = (d_{ij}(t))_{1 \le i,j \le n}$ be the $n \times n$ matrix where

$$d_{ij}(t) = \binom{i+j-1}{2j-i} + \left\{ \binom{i+j-1}{2j-i+1} + \binom{i+j-1}{2j-i-1} \right\} t + \binom{i+j-1}{2j-i} t^2$$

Use Binet-Cauchy theorem to obtain

Theorem

Let $n \ge 2$ be a positive integer. Then

$$\sum_{\substack{b \in \mathcal{B}_{2n+1} \\ \gamma(b) = b}} t^{U_2(b)} = \det D_n(t)$$

Problems

(i) Evaluation of the Pfaffians and determinants

(ii) How to enumerate "almost domino plane partitions"?

(iii) Is there a relation between the jeu de taquin and the involutions $\widetilde{\rho}$, $\widetilde{\gamma}$?

(iv) Many misterious symmetries (There appear A_n, A_n^{HT}, A_n^{VS} in various ways. What's the reason?)

Thank you!